THE SPECTRUM OF THE CESÀRO-HARDY OPERATOR ON THE HILBERT-PÓLYA SPACE

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ABSTRACT. By considering the spectrum of the Cesàro-Hardy operator on the Hilbert-Pólya space, we proved the Riemann hypothesis for Riemann zeta function and Dirichlet L-function.

1. INTRODUCTION

Denote $\mathbb{R}_+^\times = (0,\infty)$. Let $L^2(\mathbb{R}_+^\times)$ be the complex Hilbert space with the usual inner product, i.e.,

$$\langle f(x), g(x)\rangle = \int_0^\infty f(x)\overline{g(x)}dx.$$

The Cesàro-Hardy operator $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times)$ is defined by

$$\mathcal{C}(f)(x) = \frac{1}{x}\int_0^x f(t)dt,$$

where $f(x) \in L^2(\mathbb{R}_+^\times)$ is a locally integrable function. Then $\mathcal{C}$ is a bounded operator on $L^2(\mathbb{R}_+^\times)$ by Hardy inequality. In [1], Brown, Halmos and Shields showed that the spectrum of $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times)$ is the circle

$$\sigma(\mathcal{C}, L^2) = \{z \in \mathbb{C} : |1-z| = 1\}.$$

This result has been generalized by D. W. Boyd [2] to L^p space. If we consider the operator $\mathcal{C}-1$, then we will find that it is a unitary operator on $L^2(\mathbb{R}_+^\times)$. Thus, a well known result which says the spectrum of unitary operator is contained in the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ can be used to the operator $\mathcal{C}-1$.

Motivated by Alain Connes's spectral interpretation for the zeros of L-functions, Ralf Meyer[7] proved that the eigenvalues of the transpose D_-^t (see [8, §2.1])of the operator D_-(induced by D on some function space) acting on a nuclear Fréchet space are exactly the nontrivial zeros of $\zeta(s)$. Later, Xian-Jin Li [5] proved that every nontrivial zero of the zeta function is indeed an eigenvalue of D_-. His method has been generalized to Dirichlet L-function and L-function associated with newforms by Dongsheng Wu[9]. Liming Ge, Xian-Jin Li, Dongsheng Wu and Boqing Xue in [4] proved that the correspondence between the set of eigenvalues of D_- acting on $\mathcal{H}$ and the set of nontrivial zeros of $\zeta(s)$ is one-to-one.

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Inspired by the above results, we construct the connection between the Cesàro operator $\mathcal{C}$ and the operator D_- on a suitable space, which may be called Hilbert–Pólya space. Based on their work, the Riemann hypothesis for Riemann zeta function and Dirichlet L-function follows. In this view, the Riemann hypothesis comes from the symmetry of operator. In some way, it is similar to consider the elliptic function and its inverse function.

2. The invariant subspace of operator

Let $C^\infty(\mathbb{R}_+^\times)$ be the set of smooth complex valued functions on $\mathbb{R}_+^\times$ and $\mathbb{N}$ be the set of nonnegative integers. The following notations are from [9]:

$$\mathcal{H}_0 = \{f \in C^\infty(\mathbb{R}_+^\times) \mid \lim_{x\to\infty} x^m f^{(n)}(x) = 0 \text{ and } f^{(n)}(0) := \lim_{x\to 0^+} f^{(n)}(x) \text{ exists}, \forall m, n \in \mathbb{N}\}.$$

$$\mathcal{H}_\cap := \{f \in \mathcal{H}_0 \mid \int_0^\infty f(x)dx = 0, f(0) = 0\}.$$

$$\mathcal{H}_- := \{f \in \mathcal{H}_0 \mid f^{(n)}(0) = 0 \text{ for } n \in \mathbb{N}\}.$$

Here, the above definitions of $\mathcal{H}_\cap$ and $\mathcal{H}_-$ coincide with Meyer's original construction (see [9, §1.2]). By L'Hôspital's rule, we have

$$\lim_{x\to 0^+} x^{-m} f^{(n)}(x) = 0, \quad \forall m, n \in \mathbb{N},\ \forall f(x) \in \mathcal{H}_-. \tag{2.1}$$

Let χ be a primitive Dirichlet character. Define

$$\mathcal{H}_\cap^\chi := \{f \in \mathcal{H}_0 \mid f^{(2n+1)}(0) = 0 \text{ if } \chi(-1) = 1, f^{(2n)}(0) = 0 \text{ if } \chi(-1) = -1, \forall n \in \mathbb{N}\}.$$

Since $\mathcal{H}_0$ is a subspace of $L^2(\mathbb{R}_+^\times)$, $\mathcal{H}_0$ is a unitary space, i.e., a complex space with inner product. We define two operators $\mathcal{D}, \mathcal{M}$ on $\mathcal{H}_0$ by

$$\mathcal{D}f(x) = -f'(x), \quad \mathcal{M}f(x) = xf(x).$$

It is easy to check that

$$\mathcal{M}\mathcal{D} - \mathcal{D}\mathcal{M} = 1. \tag{2.2}$$

Theorem 2.1. *$\mathcal{M}\mathcal{D}$ and $-\mathcal{D}\mathcal{M}$ are invertible operators on $\mathcal{H}_-$. The inverse of $-\mathcal{D}\mathcal{M}$ is the Cesàro-Hardy operator $\mathcal{C}$.*

Proof. First, we check that the operators $\mathcal{D}$ and $\mathcal{M}$ are invertible operators on $\mathcal{H}_-$. Since

$$\mathcal{D}^{-1}f(x) = -\int_0^x f(t)dt, \quad \mathcal{M}^{-1}f(x) = \frac{f(x)}{x},$$

by higher derivative law, we have

$$(\mathcal{D}^{-1}f)^{(n)}(x) = -f^{(n-1)}(x)$$

$$(\mathcal{M}^{-1}f)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{-k} f^{(n-k)}(x). \tag{2.3}$$

From the definition of $\mathcal{H}_0$ and the equation(2.1)(2.3), we have

$$\mathcal{D}^{-1}f(x), \mathcal{M}^{-1}f(x) \in \mathcal{H}_-, \quad \forall f(x) \in \mathcal{H}_-.$$

Similarly, it is easy to see

$$\mathcal{D}f(x), \mathcal{M}f(x) \in \mathcal{H}_-, \quad \forall f(x) \in \mathcal{H}_-.$$

Hence, there are

$$(-\mathcal{D}\mathcal{M})^{-1} = \mathcal{M}^{-1}(-\mathcal{D})^{-1}.$$

We have

$$\mathcal{M}^{-1}(-\mathcal{D})^{-1}f(x) = \mathcal{M}^{-1}\int_0^x f(t)dt = \frac{1}{x}\int_0^x f(t)dt = \mathcal{C}f(x),$$

that is, $\mathcal{M}^{-1}(-\mathcal{D})^{-1} = \mathcal{C}$ on $\mathcal{H}_-$. □

For $f \in \mathcal{H}_\cap$, define the operator $\mathcal{Z}$ by

$$(\mathcal{Z}f)(x) = \sum_{n=1}^{\infty} f(nx)$$

and for $f \in \mathcal{H}_\cap^\chi$, define the operator $\mathcal{Z}_\chi$ by

$$(\mathcal{Z}_\chi f)(x) = \sum_{n=1}^{\infty} \chi(n) f(nx).$$

Then we have $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi\mathcal{H}_\cap^\chi \subset \mathcal{H}_-$(see [9, Thm.2.9], [7, Thm.3.3], §6 in [7]).

Proposition 2.2. *The spaces $\mathcal{Z}\mathcal{H}_\cap$ is invariant subspace of the operator $-\mathcal{D}\mathcal{M}$*

Proof. Take $g(x) \in \mathcal{H}_\cap$. Then we check that $g_1(x) := xg'(x) \in \mathcal{H}_\cap$. It is easy to see

$$\int_0^\infty g_1(x)dx = 0, \quad g_1(0) = 0.$$

So, $g_1(x) = xg'(x) \in \mathcal{H}_\cap$. Since

$$\begin{aligned}
-\mathcal{D}\mathcal{M}(\mathcal{Z}g(x)) &= -\mathcal{D}\mathcal{M}(\sum_{n=1}^{\infty} g(nx)) \\
&= \sum_{n=1}^{\infty} g(nx) + \sum_{n=1}^{\infty} nxg'(nx) \\
&= \mathcal{Z}(g(x)) + \mathcal{Z}(xg'(x))
\end{aligned}$$

we have $-\mathcal{D}\mathcal{M}(\mathcal{Z}g(x)) \in \mathcal{Z}\mathcal{H}_\cap$, i.e., the action of $-\mathcal{D}\mathcal{M}$ is closed on $\mathcal{Z}\mathcal{H}_\cap$. Similarly, the action of $-\mathcal{D}\mathcal{M}$ is closed on $\mathcal{Z}_\chi\mathcal{H}_\cap^\chi$. □

Denote

$$\eta(x) = x^2(\pi x^2 - \frac{3}{2})e^{-\pi x^2}, \quad \text{for } \zeta(s);$$

$$\eta_\chi(x) = \begin{cases} e^{-\pi x^2}, & \text{for } L(\chi, s) \text{ when } \chi(-1) = 1 \\ xe^{-\pi x^2}, & \text{for } L(\chi, s) \text{ when } \chi(-1) = -1. \end{cases}$$

Then we have $\mathcal{Z}\eta, \mathcal{Z}_\chi\eta_\chi \in \mathcal{H}_-$.

Definition 2.3. Define the subspace of $\mathcal{H}_-$ by

$$\bigvee\nolimits_{\mathcal{C}^{-\infty}} \mathcal{Z}\eta := \left\{ \sum_{k=-m}^{n} a_k \mathcal{C}^k \mathcal{Z}\eta : \ a_k \in \mathbb{C}, \ m, n \in \mathbb{N} \right\}.$$

$$\bigvee\nolimits_{\mathcal{C}^{-\infty}} \mathcal{Z}_\chi\eta_\chi := \left\{ \sum_{k=-m}^{n} a_k \mathcal{C}^k \mathcal{Z}_\chi\eta_\chi : \ a_k \in \mathbb{C}, \ m, n \in \mathbb{N} \right\}.$$

Then we have

Theorem 2.4. *$\bigvee_{\mathcal{C}^{-\infty}} \mathcal{Z}\eta$ and $\bigvee_{\mathcal{C}^{-\infty}} \mathcal{Z}_\chi \eta_\chi$ are invariant spaces of $\mathcal{C}$ and $\mathcal{C}^{-1} = -\mathcal{D}\mathcal{M}$. Moreover, $\mathcal{C}$ is invertible operator and its inverse operator is $-\mathcal{D}\mathcal{M}$ on $\bigvee_{\mathcal{C}^{-\infty}} \mathcal{Z}\eta$ and $\bigvee_{\mathcal{C}^{-\infty}} \mathcal{Z}_\chi \eta_\chi$.*

For each $j \in \mathbb{Z}$, denote

$$\bigvee_{\mathcal{C}^j} \mathcal{Z}\eta := \left\{ \sum_{k=-m}^{n} a_k \mathcal{C}^k \mathcal{Z}\eta : \ a_k \in \mathbb{C}, \ m, n \in \mathbb{N}, k \geq j \right\} \subset \mathcal{H}_-.$$

It is easy to see the above spaces are invariant space of $\mathcal{C}$, so is $\mathcal{C} - 1$. Then we have a filtration of invariant subspaces of $\mathcal{C}$ and $\mathcal{C} - 1$, i.e.,

$$\cdots \supset \bigvee_{\mathcal{C}^{-1}} \mathcal{Z}\eta \supset \bigvee_{\mathcal{C}^0} \mathcal{Z}\eta \supset \bigvee_{\mathcal{C}^1} \mathcal{Z}\eta \supset \cdots$$

The following theorem is clear.

Theorem 2.5. $\bigcap_{j=-\infty}^{+\infty} \bigvee_{\mathcal{C}^j} \mathcal{Z}\eta = 0$

Theorem 2.6. *The operators $\mathcal{C}$ and $\mathcal{Z}$ are commutative on $\mathcal{H}_-$. Similarly, $\mathcal{C}$ and $\mathcal{Z}_\chi$ are commutative.*

Proof. We show the case for $\mathcal{C}$ and $\mathcal{Z}$, the other case is similar. For $f(x) \in \mathcal{H}_-$, we have

$$\begin{aligned} \mathcal{C}\mathcal{Z}f(x) &= \frac{1}{x} \int_0^x \sum_{n=1}^{\infty} f(nt)dt \\ &= \sum_{n=1}^{\infty} \frac{1}{x} \int_0^x f(nt)dt \\ &= \sum_{n=1}^{\infty} \frac{1}{x} \int_0^{nx} f(t)d\frac{t}{n} \\ &= \sum_{n=1}^{\infty} \frac{1}{nx} \int_0^{nx} f(t)dt \\ &= \mathcal{Z}\mathcal{C}f(x). \end{aligned}$$

Hence, $\mathcal{C}$ and $\mathcal{Z}$ are commutative. □

Theorem 2.7. *$\bigvee_{\mathcal{C}^1} \eta \subset \mathcal{H}_\cap$, where*

$$\bigvee_{\mathcal{C}^1} \eta = \left\{ \sum_{k=1}^{n} a_k \mathcal{C}^k \eta : \ a_k \in \mathbb{C}, \ n \in \mathbb{N} \right\}$$

Moreover, $\mathcal{C}\mathcal{H}_\cap \subseteq \mathcal{H}_\cap$, i.e., $\mathcal{H}_\cap$ is an invariant space of $\mathcal{C}$. Similarly, $\mathcal{H}_\cap^\chi$ is also an invariant space of $\mathcal{C}$

Proof. By Theorem2.6, we have

$$\mathcal{Z}\left(\bigvee_{\mathcal{C}^1} \eta\right) = \bigvee_{\mathcal{C}^1} \mathcal{Z}\eta \subset \mathcal{H}_-.$$

Thus the theorem follows by [7, Thm.3.3]. The last statement is true by the similar discussion and Theorems in §6 in [7]. □

3. The operator $\mathcal{C}-1$ and Hilbert-Pólya space

Let

$$\overline{\mathcal{H}_-},\ \overline{\mathcal{Z}\mathcal{H}_\cap},\ \overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}\subset L^2(\mathbb{R}_+^\times)$$

be the closure of $\mathcal{H}_-,\mathcal{Z}\mathcal{H}_\cap,\mathcal{Z}_\chi\mathcal{H}_\cap^\chi$. Then $\overline{\mathcal{H}_-},\ \overline{\mathcal{Z}\mathcal{H}_\cap},\ \overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}$ are Hilbert space(see [3, §3.6]). Let $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$ (resp. $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}^\perp$) be the orthogonal complement of $\overline{\mathcal{Z}\mathcal{H}_\cap}$ (resp. $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}$)in $\overline{\mathcal{H}_-}$. Then we have the canonical decomposition(see [3, Thm.3.6.6])

$$\overline{\mathcal{H}_-}=\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp\oplus\overline{\mathcal{Z}\mathcal{H}_\cap}=\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}^\perp\oplus\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}.$$

Theorem 3.1. *The action of Cesàro-Hardy operator $\mathcal{C}$ on $\overline{\mathcal{H}_-},\overline{\mathcal{Z}\mathcal{H}_\cap}$ and $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}$ is closed. In other words, the above spaces are invariant subspaces of $\mathcal{C}$.*

Proof. From Theorem2.1, we know the action of $\mathcal{C}$ is closed on $\mathcal{H}_-$. Let $\{f_n\}$ be a convergent sequence in $\overline{\mathcal{H}_-}$, where $f_n\in\mathcal{H}_-$. Then we can denote $\lim\limits_{n\to\infty}f_n=f\in\overline{\mathcal{H}_-}$. Since $\mathcal{C}$ is bounded on $L^2(\mathbb{R})_+^\times$, i.e., continuous, we have $\mathcal{C}f=\lim\limits_{n\to\infty}\mathcal{C}f_n\in\overline{\mathcal{H}_-}$ under norm topology. Similarly, the action of $\mathcal{C}$ is closed on $\overline{\mathcal{Z}\mathcal{H}_\cap}$ and $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}$. □

Let $\mathcal{C}^*$ be the adjoint operator of $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times)$. Then

Lemma 3.2. *For $f(x)\in\mathcal{H}_0$, we have*

$$\mathcal{C}^*f(x)=\int_x^\infty\frac{f(t)}{t}dt$$

Proof. For each $f,g\in\mathcal{H}_0$, there is

$$\begin{aligned}\langle\mathcal{C}f,g\rangle&=\int_0^\infty\frac{1}{x}\int_0^x f(t)dt\cdot\overline{g(x)}dx\\&=\int_0^\infty\int_0^x f(t)dt\cdot\overline{\left(\frac{g(x)}{x}\right)}dx\\&=-\int_0^\infty\int_0^x f(t)dtd\int_x^\infty\overline{\left(\frac{g(t)}{t}\right)}dt\\&=-\int_0^x f(t)dt\cdot\int_x^\infty\overline{\left(\frac{g(t)}{t}\right)}dt\Big|_0^\infty+\int_0^\infty\int_x^\infty\overline{\left(\frac{g(t)}{t}\right)}dtd\int_0^x f(t)dt\\&=\int_0^\infty f(x)\int_x^\infty\overline{\left(\frac{g(t)}{t}\right)}dtdx\\&=\langle f,\mathcal{C}^*g\rangle.\end{aligned}$$

By the definition of integral, we have

$$\overline{\mathcal{C}^*g(x)}=\int_x^\infty\overline{\left(\frac{g(t)}{t}\right)}dt=\overline{\int_x^\infty\frac{g(t)}{t}dt}.$$

Hence, $\mathcal{C}^*f(x)=\int_x^\infty\frac{f(t)}{t}dt$. □

Theorem 3.3. *The operator $\mathcal{C}-1$ is unitary on $L^2(\mathbb{R}_+^\times)$*

Proof. There exist the following norm equality(see [6, Example1.6]): For $f\in L^2(\mathbb{R}_+^\times)$,

$$\|(\mathcal{C}-1)f\|=\|(\mathcal{C}^*-1)f\|=\|f\|,$$

where $\mathcal{C}^*-1=(\mathcal{C}-1)^*$ is the adjoint operator of $\mathcal{C}-1$. This means the bounded operator $\mathcal{C}-1$ and $(\mathcal{C}-1)^*$ are isometry on $L^2(\mathbb{R}_+^\times)$. By [3, Thm4.5.15], we have $(\mathcal{C}-1)^*(\mathcal{C}-1)=(\mathcal{C}-1)(\mathcal{C}-1)^*=1$. This means $\mathcal{C}-1$ is unitary. □

We have the following theorem

Theorem 3.4. *The space $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$, $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}^\perp$ are invariant spaces of the operator $\mathcal{C}-1$. Moreover, they are invariant spaces of the operator $\mathcal{C}$.*

Proof. Since the bounded operator $\mathcal{C}-1$ and $(\mathcal{C}-1)^*$ are isometry on $L^2(\mathbb{R}_+^\times)$, we have $\mathcal{C}-1$ is unitary operator on each subspace of $L^2(\mathbb{R}_+^\times)$. Hence, $\mathcal{C}-1$ is invertible operator on $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$. For every $f\in\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp, g\in\overline{\mathcal{Z}\mathcal{H}_\cap}$, we have $(\mathcal{C}-1)^{-1}g\in\overline{\mathcal{Z}\mathcal{H}_\cap}$. Hence,

$$\langle(\mathcal{C}-1)f,g\rangle=\langle f,(\mathcal{C}-1)^*g\rangle=\langle f,(\mathcal{C}-1)^{-1}g\rangle=0.$$

That is $(\mathcal{C}-1)f\in\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$ This means $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$ is an invariant subspace of the operator $\mathcal{C}-1$. Similarly, $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}^\perp$ is an invariant subspace of the operator $\mathcal{C}-1$. Since $\mathcal{C}=(\mathcal{C}-1)+1$, we have that they are invariant spaces of the operator $\mathcal{C}$. □

Definition 3.5. The space $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$ (resp. $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}^\perp$) is called Hilbert-Pólya space of the operator $\mathcal{C}-1$ with respect to Riemann zeta function(resp. Dirichlet L-function).

4. The spectrum of $\mathcal{C}-1$ on Hilbert-Pólya space

This section, we prove the Riemann hypothesis for Riemann zeta function and Dirichlet L-function, which is inspired by Meyer's paper[7], Li's result[5] and Wu's work[9].

For $f(x)\in\mathcal{H}_0$, its Mellin transform is

$$\widehat{f}(s)=\int_0^\infty f(x)x^{s-1}dx.$$

Then $\widehat{f}(s)$ admits a meromorphic extension to the whole complex plane and its only singularities are simple poles at a subset of non-positive integers(see [9, Lem2.1]). Hence, we have the derivative of $\widehat{f}(s)$ is

$$\widehat{f}^{(n)}(s)=\int_0^\infty f(x)x^{s-1}(\log x)^n dx.$$

Let ρ be a nontrivial zero of $\zeta(s)$ (resp. $L(\chi,s)$). Denote

$$\eta(x)=x^2(\pi x^2-\frac{3}{2})e^{-\pi x^2},\quad \text{for } \zeta(s);$$

$$\eta_\chi(x)=\begin{cases}e^{-\pi x^2}, & \text{for } L(\chi,s) \text{ when } \chi(-1)=1\\ xe^{-\pi x^2}, & \text{for } L(\chi,s) \text{ when } \chi(-1)=-1.\end{cases}$$

Define

$$F_\rho(x)=\begin{cases}\int_1^\infty \mathcal{Z}\eta(tx)t^{\rho-1}dt, & \text{if } \zeta(\rho)=0\\ \int_1^\infty \mathcal{Z}_\chi\eta_\chi(tx)t^{\rho-1}dt, & \text{if } L(\chi,\rho)=0.\end{cases}$$

Then we have the following theorem

Theorem 4.1. *If ρ be a nontrivial zero of $\zeta(s)$, then $F_\rho(x) \in \mathcal{H}_- \setminus \mathcal{Z}\mathcal{H}_\cap$; If ρ be a nontrivial zero of $L(\chi, s)$, then $F_\rho(x) \in \mathcal{H}_- \setminus \mathcal{Z}_\chi\mathcal{H}_\cap^\chi$. Moreover,*

$$-xF_\rho'(x) = \begin{cases} \rho F(x) + \mathcal{Z}\eta(x), & \text{if } \zeta(\rho) = 0 \\ \rho F(x) + \mathcal{Z}_\chi\eta_\chi(x), & \text{if } L(\chi,\rho) = 0. \end{cases} \tag{4.1}$$

Proof. We just prove the case for Riemann zeta function, the case for Dirichlet L-function is similar. Let ρ be a nontrivial zero of $\zeta(s)$. Take $\eta(x)$. Let

$$F_\rho(x) = \int_1^\infty \mathcal{Z}\eta(tx)t^{\rho-1}dt = x^{-\rho}\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt.$$

It is easy to see

$$F_\rho(x) = x^{-\rho}\mathcal{C}^*(x^\rho\mathcal{Z}\eta).$$

Hence,

$$\begin{aligned} -xF_\rho'(x) &= -x\left(-\rho x^{-\rho-1}\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt - x^{-\rho}\mathcal{Z}\eta(x)x^{\rho-1}\right) \\ &= \rho x^{-\rho}\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt + \mathcal{Z}\eta(x) \\ &= \rho F_\rho(x) + \mathcal{Z}\eta(x) \end{aligned}$$

Next we show that $F_\rho(x) \in \mathcal{H}_- \setminus \mathcal{Z}\mathcal{H}_\cap$.

Considering the Mellin transform of the equation(4.1), we have

$$\widehat{-xF_\rho'}(s) = \rho\widehat{F}_\rho(s) + \widehat{\mathcal{Z}\eta}(s). \tag{4.2}$$

It is easy to see

$$F_\rho(0) = \lim_{x\to0^+}\int_1^\infty \mathcal{Z}\eta(tx)t^{\rho-1}dt = \int_1^\infty \lim_{x\to0^+}\mathcal{Z}\eta(tx)t^{\rho-1}dt = 0;$$

For $\mathrm{Re}(s) > 0$, if $\mathrm{Re}(s-\rho) \le 0$, it is clear that $\lim\limits_{x\to\infty} x^sF_\rho(x) = 0$; if $\mathrm{Re}(s-\rho) > 0$, we also have

$$\begin{aligned} \lim_{x\to\infty} x^sF_\rho(x) &= \lim_{x\to\infty} x^{s-\rho}\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt \\ &= \lim_{x\to\infty}\frac{\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt}{x^{\rho-s}} \\ &= \lim_{x\to\infty}\frac{-\mathcal{Z}\eta(x)x^{\rho-1}}{(\rho-s)x^{\rho-s-1}} \\ &= \lim_{x\to\infty} -\frac{1}{\rho-s}x^s\mathcal{Z}\eta(x) \\ &= 0. \end{aligned}$$

Thus, for $\mathrm{Re}(s) > 0$,

$$\begin{aligned}\widehat{-xF'_\rho}(s) &= \int_0^\infty (-xF'_\rho(x))x^{s-1}dx \\ &= -\int_0^\infty x^s dF_\rho(x) \\ &= -x^s F_\rho(x)\Big|_0^\infty + \int_0^\infty F_\rho(x)dx^s \\ &= s\int_0^\infty F_\rho(x)x^{s-1}dx \\ &= s\widehat{F_\rho}(s)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\widehat{\mathcal{Z}\eta}(s) &= \int_0^\infty \left(\sum_{n=1}^\infty \eta(nx)\right) x^{s-1}dx \\ &= \sum_{n=1}^\infty \int_0^\infty \eta(y)\left(\frac{y}{n}\right)^{s-1} d\frac{y}{n} \\ &= \zeta(s)\widehat{\eta}(s).\end{aligned} \tag{4.4}$$

Combining equations(4.2)(4.3)(4.4) we have

$$(s-\rho)\widehat{F_\rho}(s) = \zeta(s)\widehat{\eta}(s). \tag{4.5}$$

Then $F_\rho(x) \in \mathcal{H}_-$ by [9, Corollary2.5].

Next we show that $F_\rho(x) \notin \mathcal{Z}\mathcal{H}_\cap$. This follows from [5, Thm1.1]. □

Theorem 4.2. *The spaces $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi\mathcal{H}_\cap^\chi$ are closed in $\mathcal{H}_-$. Hence,*

$$\mathcal{H}_- \setminus \mathcal{Z}\mathcal{H}_\cap = \mathcal{H}_- \setminus \overline{\mathcal{Z}\mathcal{H}_\cap}, \quad \mathcal{H}_- \setminus \mathcal{Z}_\chi\mathcal{H}_\cap^\chi = \mathcal{H}_- \setminus \overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}.$$

Proof. By [7, Thm3.3] and §6 in [7], the spaces $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi\mathcal{H}_\cap^\chi$ are closed in $\mathcal{H}_-$. The equations follow from $\mathcal{Z}\mathcal{H}_\cap = \mathcal{H}_- \cap \overline{\mathcal{Z}\mathcal{H}_\cap}$ and $\mathcal{Z}_\chi\mathcal{H}_\cap^\chi = \mathcal{H}_- \cap \overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}$. □

Theorem 4.3. *Let ρ be a nontrivial zero of $\zeta(s)$ (resp. $L(\chi, s)$). Then $\frac{1-\rho}{\rho}$ is an eigenvalue of $\mathcal{C} - 1$ on $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$ (resp. $\overline{\mathcal{Z}_\chi\mathcal{H}_\cap^\chi}^\perp$).*

Proof. We just prove the case for Riemann zeta function $\zeta(s)$. The case for Dirichlet L-function is similar. Let ρ be a nontrivial zero of $\zeta(s)$. Then $1-\rho$ is also a nontrivial zero. By Theorem4.1,4.2, there is a function $f(x) \in \mathcal{H}_- \setminus \overline{\mathcal{Z}\mathcal{H}_\cap}$ and $\eta(x) \in \mathcal{H}_\cap$ such that

$$-xf'(x) = (1-\rho)f(x) + \mathcal{Z}\eta(x).$$

Hence,

$$\rho f(x) - \mathcal{Z}\eta(x) = (xf(x))'$$

Integrating on the equation, we have

$$\int_0^x \rho f(t)dt - \int_0^x \mathcal{Z}\eta(t)dt = xf(x).$$

Dividing by ρx , we obtain

$$\mathcal{C}f(x) = \frac{1}{\rho}f(x) + \frac{1}{\rho}\mathcal{C}(\mathcal{Z}\eta(x)).$$

That is

$$(\mathcal{C}-1)f(x) = \frac{1-\rho}{\rho}f(x) + \frac{1}{\rho}\mathcal{C}(\mathcal{Z}\eta(x)). \tag{4.6}$$

Let

$$f(x) = f_1(x) + f_2(x) \tag{4.7}$$

where $f_1(x) \in \overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$, $f_2(x) \in \overline{\mathcal{Z}\mathcal{H}_\cap}$. Here $f_1(x) \neq 0$, otherwise we have $f(x) = f_2(x) \in \overline{\mathcal{Z}\mathcal{H}_\cap}$, a contradiction.

Putting this equality (4.7) in the above equation(4.6), we obtain

$$(\mathcal{C}-1)f_1(x) - \frac{1-\rho}{\rho}f_1(x) = (1-\mathcal{C})f_2(x) + \frac{1-\rho}{\rho}f_2(x) + \frac{1}{\rho}\mathcal{C}(\mathcal{Z}\eta(x)).$$

Thus $(\mathcal{C}-1)f_1(x) - \frac{1-\rho}{\rho}f_1(x) \in \overline{\mathcal{Z}\mathcal{H}_\cap}^\perp \cap \overline{\mathcal{Z}\mathcal{H}_\cap}$, we have

$$(\mathcal{C}-1)f_1(x) = \frac{1-\rho}{\rho}f_1(x).$$

Hence, $\frac{1-\rho}{\rho}$ is an eigenvalue of $\mathcal{C}-1$ on $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$. □

Theorem 4.4. *The Riemann hypothesis is true for Riemann zeta function and Dirichlet L-function.*

Proof. First, $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$ is an invariant subspace of $\mathcal{C}-1$. The eigenvalue of the operator $\mathcal{C}-1$ on $\overline{\mathcal{Z}\mathcal{H}_\cap}^\perp$ belongs to it spectrum. Since the bounded operator $\mathcal{C}-1$ is a unitary operator, its spectrum is in the circle $\{z \in \mathbb{C} : |z| = 1\}$. Therefore, the Riemann hypothesis follows by Theorem4.3. Similarly, it is true for Dirichlet L-function. □

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