Copositivity criteria of a class of fourth order 3-dimensional symmetric tensors

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Abstract

In this paper, we mainly dicuss the non-negativity conditions for quartic homogeneous polynomials with 3 variables, which is the analytic conditions of copositivity of a class of 4th order 3-dimensional symmetric tensors. For a 4th order 3-dimensional symmetric tensor with its entries 1 or -1, an analytic necessary and sufficient condition is given for its strict copositivity with the help of the properties of strictly semi-positive tensors. And by means of usual maxi-min theory, a necessary and sufficient condition is established for copositivity of such a tensor also. Applying these conclusions to a general 4th order 3-dimensional symmetric tensor, the analytic conditions are successfully obtained for verifying the (strict) copositivity, and these conditions can be very easily parsed and validated. Moreover, several (strict) inequalities of ternary quartic homogeneous polynomial are established by means of these analytic conditions.

Keywords: Analytic conditions, Copositivity, Fourth order tensors, Homogeneous polynomial

1. Introduction

One of the most direct applications of 4th order copositive tensors is to verify the vacuum stability of the Higgs scalar potential model [1, 2, 3, 4, 5, 6]. In the graph theory, the mth order copositive tensors may be directly applied to estimate the bounds on the independent number of m-uniform hypergraph [3, 7, 8, 9]. The concept of copositive tensors was introduced by Qi [10] in 2013, which is usually applied to a symmetric tensor or, more precisely, to its associated Homogeneous polynomial of degree m.

Definition 1.1. *Let $\mathcal{T} = (t_{i_1 i_2 \cdots i_m})$ be an mth order n dimensional symmetric tensor. $\mathcal{T}$ is called*

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(i) **positive semi-definite** *([11]) if m is an even number and in the Euclidean space* $\mathbb{R}^n$*, its associated Homogeneous polynomial*

$$\mathcal{T}x^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} t_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m} \geq 0;$$

(ii) **positive definite** *([11]) if m is an even number and* $\mathcal{T}x^m > 0$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$*;*

(iii) **copositive** *([10]) if* $\mathcal{T}x^m \geq 0$ *on the nonnegative orthant* $\mathbb{R}^n_+$*;*

(iv) **strictly copositive** *([10]) if* $\mathcal{T}x^m > 0$ *for all* $x \in \mathbb{R}^n_+ \setminus \{0\}$.

Clearly, the positive semi-definite tensors must be copositive, and a copositive tensor coincides with a copositive matrix if $m = 2$. The concept of copositive matrices was introduced by Motzkin [12] in 1952. Baston [13] gave an analytic way of judging copositivity of a $n \times n$ matrix in 1969.

Theorem 1.1. *(Baston [13]) Let* $M = (m_{ij})$ *be a symmetric matrix with* $|m_{ij}| = m_{ii} = 1$ *for all* $i, j \in \{1, 2, \cdots, n\}$*. Then the matrix M is copositive if and only if there is no triple* (r, s, t) *such that*

$$m_{rs} = m_{rt} = m_{st} = -1.$$

Simpson-Spector [14] and Hadeler [15] and Nadler [16] and Chang-Sederberg [17] and Andersson-Chang-Elfving [18] respectively showed the (strict) copositive conditions of 2×2 and 3×3 matrices using different methods of argumentation.

Theorem 1.2. *Let* $M = (m_{ij})$ *be a symmetric matrix. Then a* 2×2 *matrix M is (strictly) copositive if and only if*

$$m_{11} \geq 0\ (> 0), m_{22} \geq 0\ (> 0), m_{12} + \sqrt{m_{11}m_{22}} \geq 0\ (> 0);$$

a 3×3 *matrix M is (strictly) copositive if and only if for all* $i \in \{1, 2, 3\}$,

$$\begin{aligned} m_{ii} \geq 0\ (> 0), \alpha = m_{12} + \sqrt{m_{11}m_{22}} \geq 0\ (> 0), \\ \beta = m_{13} + \sqrt{m_{11}m_{33}} \geq 0\ (> 0), \gamma = m_{23} + \sqrt{m_{33}m_{22}} \geq 0\ (> 0), \\ m_{12}\sqrt{m_{33}} + m_{13}\sqrt{m_{22}} + m_{23}\sqrt{m_{11}} + \sqrt{m_{11}m_{22}m_{33}} + \sqrt{2\alpha\beta\gamma} \geq 0\ (> 0). \end{aligned}$$

Schmidt-Heβ [19] provided the nonnegative conditions of a cubic and univariate polynomial with real coefficients in non-negative real number $\mathbb{R}_+$, and Qi-Song-Zhang [20] recently gave the positivity of such a cubic polynomial, which actually gave a the (strict) copositivity of 3rd order 2-dimensional symmetric tensor (see Liu-Song [21] for more details also). Qi-Song-Zhang [20] also gave the nonnegativity and positivity of a quartic and univariate polynomial in $\mathbb{R}$, which means the positive (semi-)definitiveness of 4th order 2-dimensional tensor. Ulrich-Watson [22] and Qi-Song-Zhang [20] presented the analytic conditions of the nonnegativity of a quartic and univariate polynomial in $\mathbb{R}_+$. This actually yielded the copositivity of 4th order 2-dimensional symmetric tensor [4].

Theorem 1.3. *A 3rd order 2-dimensional symmetric tensor* $\mathcal{T} = (t_{ijk})$ *is (strictly) copositive if and only if* $t_{111} \geq 0\ (> 0)$*,* $t_{222} \geq 0\ (> 0)$*, either* $t_{112} \geq 0,\ t_{122} \geq 0$ *or*

$$4t_{111}t_{122}^3 + 4t_{112}^3t_{222} + t_{111}^2t_{222}^2 - 6t_{111}t_{112}t_{122}t_{222} - 3t_{112}^2t_{122}^2 \geq 0\ (> 0).$$

A 4th order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ *with* $t_{1111} > 0$ *and* $t_{2222} > 0$ *is copositive if and only if*

$$\begin{cases} \Delta \leq 0, t_{1222}\sqrt{t_{1111}} + t_{1112}\sqrt{t_{2222}} > 0; \\ t_{1222} \geq 0, t_{1112} \geq 0, 3t_{1122} + \sqrt{t_{1111}t_{2222}} \geq 0; \\ \Delta \geq 0, \\ |t_{1112}\sqrt{t_{2222}} - t_{1222}\sqrt{t_{1111}}| \leq \sqrt{6t_{1111}t_{1122}t_{2222} + 2t_{1111}t_{2222}\sqrt{t_{1111}t_{2222}}} \\ (i)\ -\sqrt{t_{1111}t_{2222}} \leq 3t_{1122} \leq 3\sqrt{t_{1111}t_{2222}}; \\ (ii)\ t_{1122} > \sqrt{t_{1111}t_{2222}} \text{ and} \\ t_{1112}\sqrt{t_{2222}} + t_{1222}\sqrt{t_{1111}} \geq -\sqrt{6t_{1111}t_{1122}t_{2222} - 2t_{1111}t_{2222}\sqrt{t_{1111}t_{2222}}}, \end{cases}$$

where $\Delta = 4 \times 12^3(t_{1111}t_{2222} - 4t_{1112}t_{1222} + 3t_{1122}^2)^3 - 72^2 \times 6^2(t_{1111}t_{1122}t_{2222} + 2t_{1112}t_{1122}t_{1222} - t_{1122}^3 - t_{1112}^2t_{2222} - t_{1111}t_{1222}^2)^2$.

For a special 4th order 3-dimensional tensor given by the particle physical model, Qi-Song-Zhang [23] presented a necessary and sufficient condition of copositivity, and Song-Li [4] provided an analytic condition of its copositivity. However, an analytic necessary and sufficient condition has not been found for the copositivity of a general 3-dimensional higher order tensor ($m > 2$). Even for a general 2-dimensional higher order tensor ($m > 3$), people still has not found the analytic conditions of strict copositivity untill now.

For checking the copositivity of symmetric tensors, various numerical algorithms have been employed. For example, Chen-Huang-Qi [24, 3] gave the detection algorithms based on simplicial partition; Li-Zhang-Huang-Qi [25] used an SDP relaxation algorithm; Nie-Yang-Zhang [7] devised a complete semi-definite algorithms. Taking advantage of the properties of a tensor itself, the copositivity can be described qualitatively. Song-Qi [26] showed a necessary and sufficient condition of copositivity by means of the principal sub-tensors; Song-Qi [27] applied the sign of its Pareto H-eigenvalue (Z-eigenvalue) to test the copositivity. Song-Qi [28] proved the equivalence of (strict) copositivity and (strict) semi-positivity of a symmetric tensor.

Theorem 1.4. *(Song-Qi [28]) Let* $\mathcal{T} = (t_{i_1i_2\cdots i_m})$ *be a symmetric tensor. Then* $\mathcal{T}$ *is (strictly) copositive if and only if* $\mathcal{T}$ *is (strictly) semi-positive, i.e., for each* $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n_+ \setminus \{0\}$, *there exists* $k \in \{1, 2, \cdots, n\}$ *such that*

$$x_k > 0 \text{ and } (\mathcal{T}x^{m-1})_k = \sum_{i_2\cdots i_m=1}^{n} t_{ki_2\cdots i_m}x_{i_2}\cdots x_{i_m} \geq 0\ (> 0).$$

For more details of the copositivity of a higher order tensor and a matrix, see [8, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Since there are strong connection between the semi-positivity of a tensor and the tensor complementarity problems [26, 38, 39, 40], so the copositivity of a symmetric tensor may be verified by solving the corresponding tensor complementarity problems. For more details about the tensor complementarity problems and its applications, see Refs. [41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59].

Motivation to checking the copositivity of a higher order tensor, we mainly dicuss analytic necessary and sufficient conditions of copositivity of a class of 4th order 3-dimensional symmetric tensors in this paper. With the help of Theorem 1.4, we first promote Theorem 1.1 to ones of

4th order 3-dimensional symmetric tensors, which gives an analytic necessary and sufficient condition of strict copositivity of such a class of tensors (Theorem 3.4). Secondly, we present an analytic necessary and sufficient condition of copositivity of such a class of tensors (Theorem 3.8). Finally, applying Theorems 3.4 and 3.8, the analytic sufficient conditions (Theorems 3.9, 3.10 and 3.11) are successfully proved for (strict) copositivity of a general 4th order 3-dimensional symmetric tensor. Furthermore, several (strict) inequalities of ternary quartic homogeneous polynomial (Theorems 3.12, 3.13, 3.14 and **??**) are built with the help of the argument procedure of Theorems 3.4 and 3.8.

2. Copositivity of 4th order 2-dimensional symmetric tensors

Let $T = (t_{i_1\cdots i_m})(i_j = 1,2,\dots,n, j = 1,2,\dots,m)$ be a mth-order n-dimensional symmetric tensor. Then for $x = (x_1.x_2,\cdots,x_n)^\top \in \mathbb{R}^n$, we write

$$Tx^m = x^\top(Tx^{m-1}) = \sum_{i_1\cdots i_m=1}^{n} t_{i_1\cdots i_m} x_{i_1}\cdots x_{i_m},$$

and $Tx^{m-1} = (y_1,y_2,\cdots,y_n)^\top$ is a vector with its components

$$y_k = (Tx^{m-1})_k = \sum_{i_2\cdots i_m=1}^{n} t_{ki_2\cdots i_m} x_{i_2}\cdots x_{i_m},\ k = 1,2,\cdots,n.$$

Let $f(x_1,x_2)$ be a quartic homogeneous real polynomial about two variables x_1,x_2,

$$f(x_1,x_2) = x_1^4 + 4ax_1^3x_2 + 6bx_1^2x_2^2 + 4cx_1x_2^3 + x_2^4. \tag{2.1}$$

Then it gives a 4th-order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ with its entiries,

$$t_{1111} = 1, t_{1112} = a, t_{1122} = b, t_{1222} = c, t_{2222} = 1. \tag{2.2}$$

By Theorem 1.3, the following lemma can be obtained easily.

Lemma 2.1. *Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor given by (2.2). Then $\mathcal{T}$ is copositive, i.e., $f(x_1,x_2) \geq 0$ for all $x = (x_1,x_2)^\top \geq 0$ if and only if*

(1) $\Delta' \leq 0$ *and* $a + c > 0$;

(2) $a \geq 0$, $c \geq 0$ *and* $1 + 3b \geq 0$;

(3) $\Delta' \geq 0$, $|a - c| \leq \sqrt{6b+2}$ *and (i)* $-1 \leq 3b \leq 3$, *(ii)* $b > 1$ *and* $a + c \geq -\sqrt{6b-2}$,

where $\Delta = 4 \times 12^3\Delta'$, $\Delta' = (1 - 4ac + 3b^2)^3 - 27(b + 2abc - b^3 - c^2 - a^2)^2$.

Lemma 2.2. *Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $|t_{ijkl}| = 1$ and $t_{1111} = t_{2222} = 1$. Then $\mathcal{T}$ is copositive if and only if either*

$$b = t_{1122} = 1 \text{ or } a = t_{1112} = t_{1222} = c = 1.$$

Proof. It follows from Lemma 2.1 that $\mathcal{T}$ is copositive if and only if

(1) $\Delta' \leq 0$ and $a + c > 0$;

(2) $a \geq 0, c \geq 0$ and $1 + 3b \geq 0$;

(3) $\Delta' \geq 0, |a - c| \leq \sqrt{6b + 2}$ and $-1 \leq 3b \leq 3$;.

Since $|t_{ijkl}| = 1$, then the conditions (1)-(3) mean

(1) $\Delta' \leq 0$ and $a = c = 1 \Leftrightarrow a = c = 1$ and either

$$b = 1, \Delta' = (1 - 4 + 3)^3 - 27(1 + 2 - 1^3 - 1 - 1)^2 = 0,$$

or

$$b = -1, \Delta' = (1 - 4 + 3)^3 - 27(-1 - 2 + 1 - 1 - 1)^2 < 0;$$

(2) $a = c = 1$ and $b = 1$;

(3) $\Delta' \geq 0, |a - c| \leq \sqrt{6b + 2}$ and $b = 1 \Leftrightarrow b = 1$ and either $ac = 1$,

$$\Delta' = (1 - 4 + 3)^3 - 27(1 + 2 - 1 - 1 - 1)^2 = 0, |a - c| = 0 < \sqrt{6b + 2} = \sqrt{8};$$

or $ac = -1$,

$$\Delta' = (1 + 4 + 3)^3 - 27(1 - 2 - 1 - 1 - 1)^2 > 0, |a - c| = 2 < \sqrt{6b + 2} = \sqrt{8}.$$

So the conditions (1)-(3) are equivalent to

$$b = 1 \text{ or } a = c = 1.$$

This completes the proof.

Corollary 2.1. *Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $t_{1111} \geq 1$ and $t_{2222} \geq 1$. Then*

(1) $\mathcal{T}$ is strictly copositive if

$$t_{1112} \geq 1, t_{1222} \geq 1, t_{1122} \geq -1;$$

(2) $\mathcal{T}$ is copositive if

$$t_{1112} \geq -1, t_{1222} \geq -1, t_{1122} \geq 1.$$

Proof. Let $x = (x_1, x_2)^\top \geq 0$. Then

$$\mathcal{T}x^4 = t_{1111}x_1^4 + 4t_{1112}x_1^3x_2 + 6t_{1122}x_1^2x_2^2 + 4t_{1222}x_1x_2^3 + t_{2222}x_2^4.$$

(1) It is obvious that

$$\begin{aligned}\mathcal{T}x^4 \geq& x_1^4 + 4x_1^3x_2 - 6x_1^2x_2^2 + 4x_1x_2^3 + x_2^4 = (x_1^2 + x_2^2)^2 + 4x_1x_2(x_1^2 - 2x_1x_2 + x_2^2)\\ =&(x_1^2 + x_2^2)^2 + 4x_1x_2(x_1 - x_2)^2.\end{aligned}$$

So $\mathcal{T}x^4 > 0$ for $x \geq 0$ and $x \neq 0$. Suppose not, then there exists $x = (x_1, x_2)^\top \neq 0$ such that $\mathcal{T}x^4 = 0$, and hence,

$$0 = \mathcal{T}x^4 \geq (x_1^2 + x_2^2)^2 + 4x_1x_2(x_1 - x_2)^2 \geq 0.$$

That is,

$$(x_1^2+x_2^2)^2+4x_1x_2(x_1-x_2)^2=0,$$

which means $x_1^2+x_2^2=0$, i.e., $x_1=x_2=0$, a contradiction. Therefore, $\mathcal{T}$ is strictly copositive.

(2) For any $x\geq 0$, it follows from Lemma 2.2 that

$$\mathcal{T}x^4\geq x_1^4-4x_1^3x_2+6x_1^2x_2^2-4x_1x_2^3+x_2^4=(x_1-x_2)^4\geq 0.$$

So, $\mathcal{T}$ is copositive. This completes the proof.

For a 4th-order 2-dimensional symmetric tensor $\mathcal{A}=(a_{ijkl})$ with its entires $a_{1111}>0$ and $a_{2222}>0$, let $t_{1111}=t_{2222}=1$ and

$$t_{1112}=a_{1112}a_{1111}^{-\frac{3}{4}}a_{2222}^{-\frac{1}{4}},\ t_{1122}=a_{1122}a_{1111}^{-\frac{1}{2}}a_{2222}^{-\frac{1}{2}},\ t_{1222}=a_{1222}a_{1111}^{-\frac{1}{4}}a_{2222}^{-\frac{3}{4}}.$$

For $y=(y_1,y_2)^\top$ and $x=(x_1,x_2)^\top=(a_{1111}^{\frac{1}{4}}y_1,a_{2222}^{\frac{1}{4}}y_2)^\top$, then

$$\begin{aligned}\mathcal{A}y^4=&a_{1111}y_1^4+4a_{1112}y_1^3y_2+6a_{1122}y_1^2y_2^2+4a_{1222}y_1y_2^3+a_{2222}y_2^4\\=&x_1^4+4t_{1112}x_1^3x_2+6t_{1122}x_1^2x_2^2+4t_{1222}x_1x_2^3+x_2^4=\mathcal{T}x^4.\end{aligned}$$

Obviously, the copositivity of $\mathcal{A}=(a_{ijkl})$ coincides with one of $\mathcal{T}=(t_{ijkl})$, and hence, we can establish an analytically sufficient condition of the (strict) copositivity of a gerenal 4th-order 2-dimensional symmetric tensor $\mathcal{A}=(a_{ijkl})$, that can be very easily parsed and validated.

Corollary 2.2. *Let $\mathcal{A}=(a_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $a_{1111}>0$ and $a_{2222}>0$. Then*

(1) $\mathcal{A}$ is strictly copositive if

$$a_{1112}\geq a_{1111}^{\frac{3}{4}}a_{2222}^{\frac{1}{4}},\ a_{1122}\geq-\sqrt{a_{1111}a_{2222}},\ a_{1222}\geq a_{1111}^{\frac{1}{4}}a_{2222}^{\frac{3}{4}};$$

(2) $\mathcal{A}$ is copositive if

$$a_{1112}\geq -a_{1111}^{\frac{3}{4}}a_{2222}^{\frac{1}{4}},\ a_{1122}\geq \sqrt{a_{1111}a_{2222}},\ a_{1222}\geq -a_{1111}^{\frac{1}{4}}a_{2222}^{\frac{3}{4}}.$$

3. Copositivity of 4th order 3-dimensional symmetric tensors

3.1. Analytical expressions of strict copositivity

Theorem 3.1. Let $\mathcal{T}=(t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}|=t_{iiii}=t_{iiij}=1$ for all $i,j,k,l\in\{1,2,3\}$. If there is at most one -1 in $\{t_{1123},t_{1223},t_{1233}\}$, then $\mathcal{T}$ is strictly copositive.

Proof. Without loss the generality, let $t_{1123}=-1, t_{1223}=t_{1233}=1$. Then

$$\begin{aligned}\mathcal{T}x^4=&x_1^4+x_2^4+x_3^4+6t_{1122}x_1^2x_2^2+6t_{1133}x_1^2x_3^2+6t_{2233}x_2^2x_3^2+4x_1^3x_2+4x_1^3x_3+4x_1x_2^3+4x_1x_3^3\\&+4x_2^3x_3+4x_2x_3^3-12x_1^2x_2x_3+12x_1x_2^2x_3+12x_1x_2x_3^2,\end{aligned}$$

and so,

$$\mathcal{T}x^3 = \frac{1}{4}\nabla\mathcal{T}x^4 = \begin{pmatrix} \sum\limits_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l \\ \sum\limits_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l \\ \sum\limits_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l \end{pmatrix}$$

It follows from Theorem 1.4 that we need only show that $\mathcal{T} = (t_{ijkl})$ is strictly semi-positive, i.e., for $x = (x_1, x_2, x_3)^\top \geq 0$, there exists $k \in \{1, 2, 3\}$ such that

$$x_k > 0 \text{ and } (\mathcal{T}x^3)_k > 0.$$

For each $i \in \{1, 2, 3\}$, we have

$$\begin{aligned}
(\mathcal{T}x^3)_1 = &\sum_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l = x_1^3 + x_2^3 + x_3^3 + 3t_{1122}x_1x_2^2 + 3t_{1133}x_1x_3^2 + 3x_1^2x_2 + 3x_1^2x_3 \\
&+ 3x_2^2x_3 + 3x_2x_3^2 - 6x_1x_2x_3 \\
\geq &x_1^3 + x_2^3 + x_3^3 - 3x_1x_2^2 - 3x_1x_3^2 + 3x_1^2x_2 + 3x_1^2x_3 + 3x_2^2x_3 + 3x_2x_3^2 - 6x_1x_2x_3 \\
= &(x_2 + x_3 - x_1)^3 + 2x_1^3 = x_1^3 + (x_2 + x_3 - x_1 + x_1)((x_2 + x_3 - x_1)^2 - x_1(x_2 + x_3 - x_1) + x_1^2) \\
= &x_1^3 + (x_2 + x_3)((x_2 + x_3 - x_1)^2 - x_1(x_2 + x_3 - x_1) + x_1^2);
\end{aligned}$$

$$\begin{aligned}
(\mathcal{T}x^3)_2 = &\sum_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l = x_1^3 + x_2^3 + x_3^3 + 3t_{1122}x_1^2x_2 + 3t_{2233}x_2x_3^2 + 3x_1x_2^2 + 3x_2^2x_3 \\
&- 3x_1^2x_3 + 3x_1x_3^2 + 6x_1x_2x_3 \\
\geq &x_1^3 + x_2^3 + x_3^3 - 3x_1^2x_2 - 3x_2x_3^2 + 3x_1x_2^2 + 3x_2^2x_3 - 3x_1^2x_3 + 3x_1x_3^2 + 6x_1x_2x_3 \\
= &(x_1 + x_3 - x_2)^3 + 2x_2^3 + 12x_1x_2x_3 - 6x_1^2x_3 = ((x_1 + x_3 - x_2)^3 + 2x_2^3) + 6x_1x_3(2x_2 - x_1);
\end{aligned}$$

$$\begin{aligned}
(\mathcal{T}x^3)_3 = &\sum_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l = x_1^3 + x_2^3 + x_3^3 + 3t_{1133}x_1^2x_3 + 3t_{2233}x_2^2x_3 + 3x_1x_3^2 + 3x_2x_3^2 \\
&- 3x_1^2x_2 + 3x_1x_2^2 + 6x_1x_2x_3 \\
\geq &x_1^3 + x_2^3 + x_3^3 - 3x_1^2x_3 - 3x_2^2x_3 + 3x_1x_3^2 + 3x_2x_3^2 - 3x_1^2x_2 + 3x_1x_2^2 + 6x_1x_2x_3 \\
= &(x_1 + x_2 - x_3)^3 + 2x_3^3 + 12x_1x_2x_3 - 6x_1^2x_2 = ((x_1 + x_2 - x_3)^3 + 2x_3^3) + 6x_1x_2(2x_3 - x_1).
\end{aligned}$$

So, it follows that

- $x_1 > 0$, $(\mathcal{T}x^3)_1 > 0$, which is done; otherwise,
- $x_1 = 0, x_2 > 0$, $(\mathcal{T}x^3)_2 > 0$;
- $x_1 = 0, x_3 > 0$, $(\mathcal{T}x^3)_3 > 0$,

and hence, $\mathcal{T}$ is strictly copositive.

We definite

$$\mathcal{T}' = (t'_{ijkl}) \le \mathcal{T} = (t_{ijkl}) \Leftrightarrow t'_{ijkl} \le t_{ijkl}, \text{ for all } i, j, k, l.$$

Then for all $x \in \mathbb{R}^n_+$, we have

$$\mathcal{T}' x^4 = \sum_{i,j,k,l=1}^{n} t'_{ijkl} x_i x_j x_k x_l \le \sum_{i,j,k,l=1}^{n} t_{ijkl} x_i x_j x_k x_l = \mathcal{T} x^4.$$

So, the (strict) copositivity of a tensor $\mathcal{T}'$ implies one of $\mathcal{T}$. Therefore, from Theorem 3.1, the following conclusions are obvious.

Corollary 3.2. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor. If $t_{iiii} \ge 1, t_{iiij} \ge 1, t_{iijj} \ge -1$ for all $i, j \in \{1, 2, 3\}, i \ne j$ and one of the following conditions,

(1) $t_{1123} \ge -1, t_{1223} \ge 1, t_{1233} \ge 1$;

(2) $t_{1123} \ge 1, t_{1223} \ge -1, t_{1233} \ge 1$;

(3) $t_{1123} \ge 1, t_{1223} \ge 1, t_{1233} \ge -1$,

then $\mathcal{T}$ is strictly copositive.

Theorem 3.3. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iiij} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. Then $\mathcal{T}$ is srictly copositive if and only if

(1) $t_{1122} = t_{1133} = t_{2233} = 1$ if $t_{1123} = t_{1223} = t_{1233} = -1$;

(2) there is at least one 1 in $\{t_{1122}, t_{1133}, t_{2233}\}$ if two of $\{t_{1123}, t_{1223}, t_{1233}\}$ are only -1.

Proof. Necessity. If $\mathcal{T}$ is srictly copositive, but the conditions don't hold. Then for $x = (1, 1, 1)^\top$, it follows that (1) one of $\{t_{1122}, t_{1133}, t_{2233}\}$ is -1,

$$\begin{aligned}\mathcal{T} x^4 =& x_1^4 + x_2^4 + x_3^4 + 6t_{1122}x_1^2x_2^2 + 6t_{1133}x_1^2x_3^2 + 6t_{2233}x_2^2x_3^2 + 4x_1^3x_2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 \\ &+ 4x_2^3x_3 + 4x_2x_3^3 + 12t_{1123}x_1^2x_2x_3 + 12t_{1223}x_1x_2^2x_3 + 12t_{1233}x_1x_2x_3^2 \\ =& 27 + 12 - 6 - 36 = -3 < 0;\end{aligned}$$

(2) $t_{1122} = t_{1133} = t_{2233} = -1$,

$$\mathcal{T} x^4 = \sum_{i,j,k,l=1}^{3} t_{ijkl} x_i x_j x_k x_l = 27 - 18 - 24 + 12 = -3 < 0.$$

So, $\mathcal{T}$ is not srictly copositive.

Sufficiency. (1) Since $t_{1122} = t_{1133} = t_{2233} = 1$ and $t_{1123} = t_{1223} = t_{1233} = -1$, then

$$\begin{aligned}\mathcal{T} x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 4x_1^3x_2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 \\ &- 12x_1^2x_2x_3 - 12x_1x_2^2x_3 - 12x_1x_2x_3^2,\end{aligned}$$

and so,

$$
\begin{aligned}
(\mathcal{T}x^3)_1 = \sum_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3x_1x_2^2 + 3x_1x_3^2 + 3x_1^2x_2 + 3x_1^2x_3 \\
&- 3x_2^2x_3 - 3x_2x_3^2 - 6x_1x_2x_3 \\
&=(x_2 + x_3 - x_1)^3 + 2x_1^3 + 6x_1x_2^2 + 6x_1x_3^2 - 6x_2^2x_3 - 6x_2x_3^2 \\
&=(x_2 + x_3 - x_1)^3 + 2x_1^3 + 6x_2^2(x_1 - x_3) + 6x_3^2(x_1 - x_2);
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{T}x^3)_2 = \sum_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 + 3x_2x_3^2 + 3x_1x_2^2 + 3x_2^2x_3 \\
&- 3x_1^2x_3 - 6x_1x_2x_3 - 3x_1x_3^2 \\
&=(x_1 + x_3 - x_2)^3 + 2x_2^3 + 6x_1^2x_2 + 6x_2x_3^2 - 6x_1^2x_3 - 6x_1x_3^2 \\
&=(x_1 + x_3 - x_2)^3 + 2x_2^3 + 6x_1^2(x_2 - x_3) + 6x_3^2(x_2 - x_1);
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{T}x^3)_3 = \sum_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_3 + 3x_2^2x_3 + 3x_1x_3^2 \\
&+ 3x_2x_3^2 - 3x_1^2x_2 - 3x_1x_2^2 - 6x_1x_2x_3 \\
&=(x_1 + x_2 - x_3)^3 + 2x_3^3 + 6x_1^2x_3 + 6x_2^2x_3 - 6x_1^2x_2 - 6x_1x_2^2 \\
&=(x_1 + x_2 - x_3)^3 + 2x_3^3 + 6x_1^2(x_3 - x_2) + 6x_2^2(x_3 - x_1).
\end{aligned}
$$

So, it follows that

- $x_1 \geq \max\{x_2, x_3\}$, $(\mathcal{T}x^3)_1 > 0$, which is done; otherwise,
- $x_2 \geq \max\{x_1, x_3\}$, $(\mathcal{T}x^3)_2 > 0$;
- $x_3 \geq \max\{x_1, x_2\}$, $(\mathcal{T}x^3)_3 > 0$;

and hence, $\mathcal{T}$ is strictly copositive.

(2) We might take $t_{1122} = t_{1233} = 1$ and $t_{1133} = t_{2233} = t_{1123} = t_{1223} = -1$, then

$$
\begin{aligned}
\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 - 6x_1^2x_3^2 - 6x_2^2x_3^2 + 4x_1^3x_2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 \\
&- 12x_1^2x_2x_3 - 12x_1x_2^2x_3 + 12x_1x_2x_3^2,
\end{aligned}
$$

and so,

$$
\begin{aligned}
(\mathcal{T}x^3)_1 = \sum_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3x_1x_2^2 - 3x_1x_3^2 + 3x_1^2x_2 + 3x_1^2x_3 \\
&- 3x_2^2x_3 + 3x_2x_3^2 - 6x_1x_2x_3 \\
&=(x_2 + x_3 - x_1)^3 + 2x_1^3 + 6x_1x_2^2 - 6x_2^2x_3 \\
&=(x_2 + x_3 - x_1)^3 + 2x_1^3 + 6x_2^2(x_1 - x_3);
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{T}x^3)_2 = \sum_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 - 3x_2x_3^2 + 3x_1x_2^2 + 3x_2^2x_3 \\
&\quad - 3x_1^2x_3 - 6x_1x_2x_3 + 3x_1x_3^2 \\
&=(x_1 + x_3 - x_2)^3 + 2x_2^3 + 6x_1^2x_2 - 6x_1^2x_3 \\
&=(x_1 + x_3 - x_2)^3 + 2x_2^3 + 6x_1^2(x_2 - x_3); \\
(\mathcal{T}x^3)_3 = \sum_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 - 3x_1^2x_3 - 3x_2^2x_3 + 3x_1x_3^2 + 3x_2x_3^2 \\
&\quad - 3x_1^2x_2 - 3x_1x_2^2 + 6x_1x_2x_3 \\
&=(x_1 + x_2 - x_3)^3 + 2x_3^3 + 12x_1x_2x_3 - 6x_1^2x_2 - 6x_1x_2^2 \\
&=(x_1 + x_2 - x_3)^3 + 2x_3^3 + 6x_1x_2((x_3 - x_2) + (x_3 - x_1)).
\end{aligned}
$$

So, it follows that

- $x_1 \geq x_3$ and $x_1 > 0$, $(\mathcal{T}x^3)_1 > 0$, which is done; otherwise,
- $x_2 \geq x_3$ and $x_2 > 0$, $(\mathcal{T}x^3)_2 > 0$;
- $x_3 \geq \max\{x_1, x_2\}$, $(\mathcal{T}x^3)_3 > 0$;

and hence, $\mathcal{T}$ is strictly copositive.

Combing the conclusions of Theorems 3.1 and 3.3, the main result is bulit in this subsection.

Theorem 3.4. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iiij} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. Then $\mathcal{T}$ is strictly copositive if and only if

(1) there is at most one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;

(2) two of $\{t_{1123}, t_{1223}, t_{1233}\}$ are only -1 and there is at least one 1 in $\{t_{1122}, t_{1133}, t_{2233}\}$;

(3) $t_{1123} = t_{1223} = t_{1233} = -1$ and $t_{1122} = t_{1133} = t_{2233} = 1$.

3.2. Analytical expressions of copositivity

Theorem 3.5. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iijj} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. If $t_{1123} = t_{1223} = t_{1233} = 1$, then $\mathcal{T}$ is copositive.

Proof. Without loss the generality, let $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3_+$ with $x_1 + x_2 + x_3 = 1$. Then

$$
\begin{aligned}
\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 \\
&+ 4t_{1112}x_1^3x_2 + 4t_{1113}x_1^3x_3 + 4t_{1222}x_1x_2^3 + 4t_{1333}x_1x_3^3 + 4t_{2223}x_2^3x_3 + 4t_{2333}x_2x_3^3,
\end{aligned}
$$

and so,

$$
\begin{aligned}
\mathcal{T}x^4 \geq& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 - 4x_1^3x_2 - 4x_1^3x_3 - 4x_1x_2^3 - 4x_1x_3^3 - 4x_2^3x_3 - 4x_2x_3^3 \\
&+ 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 \\
=& (x_1 + x_2 - x_3)^4 + 8(3x_1^2x_2x_3 + 3x_1x_2^2x_3 - x_1^3x_2 - x_1x_2^3).
\end{aligned}
$$

Let $f(x_1,x_2,x_3) = (x_1+x_2-x_3)^4 + 8(3x_1^2x_2x_3 + 3x_1x_2^2x_3 - x_1^3x_2 - x_1x_2^3)$. Solve the constrained optimization problem in the non-negative orthant $\mathbb{R}^3_+$:

$$\begin{aligned}&\min\ f(x_1,x_2,x_3)\\ &\text{s. t. } x_1+x_2+x_3 = 1.\end{aligned}$$

Then the function $f(x_1,x_2,x_3)$ reaches the minimum value 0 at a point $\left(\frac{1}{2},0,\frac{1}{2}\right)$ or $\left(0,\frac{1}{2},\frac{1}{2}\right)$ or $\left(\frac{1}{2},\frac{1}{2},0\right)$, and hence, $\mathcal{T}x^4 \geq f(x_1,x_2,x_3) \geq 0$ for any $x \geq 0$. That is, $\mathcal{T}$ is copositive.

From the proof of Theorem 3.5, the following conclusion is easily obtained.

Corollary 3.6. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor. If for all $i,j \in \{1,2,3\}$ and $i \neq j$,

$$t_{iiii} \geq 1, t_{iijj} \geq 1, t_{iiij} \geq -1, t_{1123} \geq 1, t_{1223} \geq 1, t_{1233} \geq 1$$

then $\mathcal{T}$ is copositive.

Theorem 3.7. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iijj} = 1$ for all $i,j,k,l \in \{1,2,3\}$. Then $\mathcal{A}$ is copositive if and only if

(1) there is at least one 1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$ if there is only one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;

(2) there is at least two 1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$ if there is only two -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;

(3) $t_{iiij} = 1, i,j = 1,2,3, i \neq j$ if $t_{1123} = t_{1223} = t_{1233} = -1$.

Proof. Necessity. Suppose $\mathcal{T}$ is copositive, but the conditions don't hold.

(1) Assume $t_{iiij} = -1$ for all $i,j = 1,2,3, i \neq j$. Then we might take $t_{1123} = -1$, and so for $x = (3,1,1)^\top$,

$$\begin{aligned}\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 - 4x_1^3x_2 - 4x_1^3x_3 - 4x_1x_2^3 - 4x_1x_3^3\\ &- 4x_2^3x_3 - 4x_2x_3^3 - 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2\\ =&(x_1+x_2-x_3)^4 + 8(3x_1x_2^2x_3 - x_1^3x_2 - x_1x_2^3)\\ =&(3+1-1)^4 + 8(3^2 - 3^3 - 3) < 0.\end{aligned}$$

(2) If there are five -1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$, then for $x = (1,1,1)^\top$,

$$\begin{aligned}\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2\\ &+ 4t_{1112}x_1^3x_2 + 4t_{1113}x_1^3x_3 + 4t_{1222}x_1x_2^3 + 4t_{1333}x_1x_3^3 + 4t_{2223}x_2^3x_3 + 4t_{2333}x_2x_3^3\\ &+ 12t_{1123}x_1^2x_2x_3 + 12t_{1223}x_1x_2^2x_3 + 12t_{1233}x_1x_2x_3^2\\ =&21 + 4 - 20 - 24 + 12 = -7 < 0;\end{aligned}$$

(3) Assume there is one -1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$, we might let $t_{1112} = -1$. Then for $x = (4,3,2)^\top$,

$$\begin{aligned}\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 - 4x_1^3x_2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3\\ &+ 4x_2^3x_3 + 4x_2x_3^3 - 12x_1^2x_2x_3 - 12x_1x_2^2x_3 - 12x_1x_2x_3^2\\ =&(x_1+x_2+x_3)^4 - 8(x_1^3x_2 + 3x_1^2x_2x_3 + 3x_1x_2^2x_3 + 3x_1x_2x_3^2)\\ =&(4+3+2)^4 - 8(4^3\times 3 + 3\times 4^2\times 3\times 2 + 3\times 4\times 3^2\times 2 + 3\times 4\times 3\times 2^2) < 0.\end{aligned}$$

So, $\mathcal{T}$ is not copositive, which is a contradiction, and hence, the conditions hold.

Sufficiency. $\mathcal{T}x^4$ may be rewritten as follows,

$$\begin{aligned}\mathcal{T}x^4 =&(x_1+x_2-x_3)^4+4(t_{1112}-1)x_1^3x_2+4(t_{1222}-1)x_1x_2^3+4(t_{1333}+1)x_1x_3^3\\&+4(t_{1113}+1)x_1^3x_3+4(t_{2223}+1)x_2^3x_3+4(t_{2333}+1)x_2x_3^3\\&+12(t_{1123}+1)x_1^2x_2x_3+12(t_{1223}+1)x_1x_2^2x_3+12(t_{1233}-1)x_1x_2x_3^2;\end{aligned}$$

$$\begin{aligned}\mathcal{T}x^4 =&(x_1-x_2+x_3)^4+4(t_{1112}+1)x_1^3x_2+4(t_{1222}+1)x_1x_2^3+4(t_{1333}-1)x_1x_3^3\\&+4(t_{1113}-1)x_1^3x_3+4(t_{2223}+1)x_2^3x_3+4(t_{2333}+1)x_2x_3^3\\&+12(t_{1123}+1)x_1^2x_2x_3+12(t_{1223}-1)x_1x_2^2x_3+12(t_{1233}+1)x_1x_2x_3^2;\end{aligned}$$

$$\begin{aligned}\mathcal{T}x^4 =&(x_1-x_2-x_3)^4+4(t_{1112}+1)x_1^3x_2+4(t_{1222}+1)x_1x_2^3+4(t_{1333}+1)x_1x_3^3\\&+4(t_{1113}+1)x_1^3x_3+4(t_{2223}-1)x_2^3x_3+4(t_{2333}-1)x_2x_3^3\\&+12(t_{1123}-1)x_1^2x_2x_3+12(t_{1223}+1)x_1x_2^2x_3+12(t_{1233}+1)x_1x_2x_3^2.\end{aligned}$$

Clearly, for the boundary points of the non-negative orthant, $x=(0,x_2,x_3)$, $(x_1,0,x_3)$, $(x_1,x_2,0)$, it follows from Lemma 2.2 that $\mathcal{T}x^4\geq 0$. Let $x_1+x_2+x_3=1$ in the sequal.

(1) Without loss the generality, let $t_{1112}=t_{1233}=t_{1223}=1$ and $t_{1113}=t_{1333}=t_{2333}=t_{1222}=t_{2223}=t_{1123}=-1$. Then

$$\mathcal{T}x^4=(x_1+x_2-x_3)^4+8(3x_1x_2^2x_3-x_1x_2^3).$$

Solve the constrained optimization problem in the non-negative orthant $\mathbb{R}^3_+$:

$$\begin{aligned}&\min\ \mathcal{T}x^4\\&\text{s. t. } x_1+x_2+x_3=1.\end{aligned}$$

Then the polynomial $\mathcal{T}x^4$ reaches the minimum value 0 at a point $\left(\frac{1}{2},0,\frac{1}{2}\right)$ or $\left(0,\frac{1}{2},\frac{1}{2}\right)$, and hence, $\mathcal{T}x^4\geq 0$ for all $x\geq 0$. That is, $\mathcal{T}$ is copositive.

(2) Without loss the generality, let $t_{1112}=t_{2223}=t_{1233}=1$ and $t_{1113}=t_{1333}=t_{2333}=t_{1222}=t_{1123}=t_{1223}=-1$. Then

$$\mathcal{T}x^4=(x_1+x_2-x_3)^4+8x_2^3(x_3-x_1).$$

Let $f(x_1,x_2,x_3,\lambda)=\mathcal{T}x^4+\lambda(x_1+x_2+x_3-1)$. Then the stationary points of the function $f(x_1,x_2,x_3,\lambda)$ are the solution to this system of equations,

$$\begin{cases}f'_{x_1}(x_1,x_2,x_3)=4(x_1+x_2-x_3)^3-8x_2^3+\lambda=0,\\f'_{x_2}(x_1,x_2,x_3)=4(x_1+x_2-x_3)^3+24x_2^2(x_3-x_1)+\lambda=0,\\f'_{x_3}(x_1,x_2,x_3)=-4(x_1+x_2-x_3)^3+8x_2^3+\lambda=0,\\x_1+x_2+x_3=1\end{cases}$$

Solve it in the non-negative orthant $\mathbb{R}^3_+$,

$$x_1=x_3=\frac{1}{2},x_2=0,\lambda=0.$$

So $\mathcal{T}x^4 \geq 0$ at the boundary points, and then $\mathcal{T}x^4 \geq 0$ for all $x \geq 0$. That is, $\mathcal{T}$ is copositive.

(3) It follows from There 3.3 (1) that $\mathcal{T}$ is copositive. This completes the proof.

Combing the conclusions of Theorems 3.5 and 3.7, the main result is established in this subsection.

Theorem 3.8. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iijj} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. Then $\mathcal{A}$ is copositive if and only if

(1) $t_{1123} = t_{1223} = t_{1233} = 1$;

(2) there is at least one 1 in $\{t_{iiij}; i, j = 1, 2, 3, i \neq j\}$ if there is only one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;

(3) there is at least two 1 in $\{t_{iiij}; i, j = 1, 2, 3, i \neq j\}$ if there is only two -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;

(4) $t_{iiij} = 1, i, j = 1, 2, 3, i \neq j$ if $t_{1123} = t_{1223} = t_{1233} = -1$.

3.3. Applications to a gerenal tensor

In this subsection, we apply Theorems 3.4 and 3.8 to find the (strict) copositivity of a gerenal 4th-order 3-dimensional symmetric tensor, and moreover, these analytic conditions can be very easily parsed and verified.

For a 4th-order 3-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ with its entires $t_{iiii} > 0$ for all $i \in \{1, 2, 3\}$, let $\mathcal{T}' = (t'_{ijkl})$ be a symmetric tensor with its entires $t'_{1111} = t'_{2222} = t'_{3333} = 1$ and

$$t'_{1112} = t_{1112}t_{1111}^{-\frac{3}{4}}t_{2222}^{-\frac{1}{4}}, t'_{1122} = t_{1122}t_{1111}^{-\frac{1}{2}}t_{2222}^{-\frac{1}{2}}, t'_{1222} = t_{1222}t_{1111}^{-\frac{1}{4}}t_{2222}^{-\frac{3}{4}},$$

$$t'_{1113} = t_{1113}t_{1111}^{-\frac{3}{4}}t_{3333}^{-\frac{1}{4}}, t'_{1133} = t_{1133}t_{1111}^{-\frac{1}{2}}t_{3333}^{-\frac{1}{2}}, t'_{1333} = t_{1333}t_{1111}^{-\frac{1}{4}}t_{3333}^{-\frac{3}{4}},$$

$$t'_{2223} = t_{2223}t_{2222}^{-\frac{3}{4}}t_{3333}^{-\frac{1}{4}}, t'_{2233} = t_{2233}t_{2222}^{-\frac{1}{2}}t_{3333}^{-\frac{1}{2}}, t'_{2333} = t_{2333}t_{2222}^{-\frac{1}{4}}t_{3333}^{-\frac{3}{4}},$$

$$t'_{1123} = t_{1123}t_{1111}^{-\frac{1}{2}}t_{2222}^{-\frac{1}{4}}t_{3333}^{-\frac{1}{4}}, t'_{1223} = t_{1223}t_{1111}^{-\frac{1}{4}}t_{2222}^{-\frac{1}{2}}t_{3333}^{-\frac{1}{4}}, t'_{1233} = t_{1233}t_{1111}^{-\frac{1}{4}}t_{2222}^{-\frac{1}{4}}t_{3333}^{-\frac{1}{2}}.$$

For $y = (y_1, y_2, y_3)^\top$ and $x = (x_1, x_2, x_3)^\top = (t_{1111}^{\frac{1}{4}}y_1, t_{2222}^{\frac{1}{4}}y_2, t_{3333}^{\frac{1}{4}}y_3)^\top$, then

$$\begin{aligned}
\mathcal{T}y^4 =& t_{1111}y_1^4 + 4t_{1112}y_1^3y_2 + 6t_{1122}y_1^2y_2^2 + 4t_{1222}y_1y_2^3 + t_{2222}y_2^4 + 4t_{1113}y_1^3y_3 + 6t_{1133}y_1^2y_3^2 \\
&+ 4t_{1333}y_1y_3^3 + t_{3333}y_3^4 + 4t_{2223}y_2^3y_3 + 6t_{2233}y_2^2y_3^2 + 4t_{2333}y_2y_3^3 \\
&+ 12t_{1123}y_1^2y_2y_3 + 12t_{1223}y_1y_2^2y_3 + 12t_{1233}y_1y_2y_3^3 \\
=& x_1^4 + 4t'_{1112}x_1^3x_2 + 6t'_{1122}x_1^2x_2^2 + 4t'_{1222}x_1x_2^3 + x_2^4 + 4t'_{1113}x_1^3x_3 + 6t'_{1133}x_1^2x_3^2 \\
&+ 4t'_{1333}x_1x_3^3 + x_3^4 + 4t'_{2223}x_2^3x_3 + 6t'_{2233}x_2^2x_3^2 + 4t'_{2333}x_2x_3^3 \\
&+ 12t'_{1123}x_1^2x_2x_3 + 12t'_{1223}x_1x_2^2x_3 + 12t'_{1233}x_1x_2x_3^2 \\
=& \mathcal{T}'x^4.
\end{aligned}$$

It is obvious that the copositivity of symmetric tensor $\mathcal{T} = (t_{ijkl})$ is equivalent to the copositivity of $\mathcal{T}' = (t'_{ijkl})$. So, applying Corollaries 3.2 and 3.6 (or Theorems 3.4 (1) and 3.8 (1) and (2) to establish easily the following conclusions .

Theorem 3.9. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $t_{iiii} > 0$ for all $i \in \{1,2,3\}$. Assume that

$$t_{1123} \geq t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}$$

Then (1) $\mathcal{T}$ is strictly copositive if for all $i, j \in \{1,2,3\}$ and $i \neq j$,

$$t_{iijj} \geq -\sqrt{t_{iiii} t_{jjjj}}, \quad t_{iiij} \geq t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}};$$

(2) $\mathcal{T}$ is copositive if for all $i, j \in \{1,2,3\}$ and $i \neq j$,

$$t_{iijj} \geq \sqrt{t_{iiii} t_{jjjj}}, \quad t_{iiij} \geq -t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}}.$$

From Theorems 3.4 (3) or 3.8 (4), the following conclusions are established easily.

Theorem 3.10. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $t_{iiii} > 0$ for all $i \in \{1,2,3\}$. Assume that

$$t_{1123} \geq -t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}.$$

Then $\mathcal{T}$ is strictly copositive if for all $i, j \in \{1,2,3\}$ and $i \neq j$,

$$t_{iijj} \geq \sqrt{t_{iiii} t_{jjjj}}, \quad t_{iiij} \geq t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}}.$$

From Theorems 3.4 (2) and 3.8 (3), the following conclusions are established easily.

Theorem 3.11. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $t_{iiii} > 0$ for all $i \in \{1,2,3\}$. Assume that one of the following three conditions holds,

(a) $t_{1123} \geq -t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}};$

(b) $t_{1123} \geq t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}};$

(c) $t_{1123} \geq -t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}.$

Then (1) $\mathcal{T}$ is strictly copositive if for all $i, j \in \{1,2,3\}$ and $i \neq j$,

$$t_{iiij} \geq t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}};$$

and there is one $t_{kkll} \in \{t_{iijj}; i, j = 1,2,3, i \neq j\}$ such that

$$t_{kkll} \geq \sqrt{t_{kkkk} t_{llll}} \text{ and } t_{iijj} \geq -\sqrt{t_{iiii} t_{jjjj}} \text{ for all } t_{iijj} \neq t_{kkll}.$$

(2) $\mathcal{T}$ is copositive if for all $i, j \in \{1,2,3\}$ and $i \neq j$,

$$t_{iijj} \geq \sqrt{t_{iiii} t_{jjjj}}$$

and there is two $t_{kkkl}, t_{sssr} \in \{t_{iiij}; i, j = 1,2,3, i \neq j\}$ such that

$$t_{kkkl} \geq t_{kkkk}^{\frac{3}{4}} t_{llll}^{\frac{1}{4}}, \ t_{sssr} \geq t_{sss}^{\frac{3}{4}} t_{rrrr}^{\frac{1}{4}}$$

and for all $t_{iiij} \in \{t_{iiij}; i, j = 1,2,3, i \neq j\} \setminus \{t_{kkkl}, t_{sssr}\}$,

$$t_{iiij} \geq -t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}}.$$

3.4. Ternary quartic inequalities

In this subsection, we apply Theorems 3.4 and 3.8 to show several ternary quartic (strict) inequalities. Applying Theorems 3.4 to establish easily the following strict inequalities .

Theorem 3.12. *If* $(x_1, x_2, x_3) \neq (0,0,0)$ *and* $x_i \geq 0,\ i = 1,2,3,$ *then*

(i) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 > 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2,$ *or equivalently,*

$$(x_1 + x_2 + x_3)^4 > 24x_1x_2x_3(x_1 + x_2 + x_3);$$

(ii) $x_1^4 + x_2^4 + x_3^4 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 > 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1^2x_2x_3,$ *or equivalently,*

$$(x_1 + x_2 + x_3)^4 > 12(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + 2x_1^2x_2x_3);$$

(iii) $x_1^4 + x_2^4 + x_3^4 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1x_2^2x_3 + 12x_1^2x_2x_3 > 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1x_2x_3^2,$ *or equivalently,*

$$(x_1 + x_2 + x_3)^4 > 12(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + 2x_1x_2x_3^2);$$

(iv) $x_1^4 + x_2^4 + x_3^4 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1^2x_2x_3 + 12x_1x_2x_3^2 > 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1x_2^2x_3,$ *or equivalently,*

$$(x_1 + x_2 + x_3)^4 > 12(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + 2x_1x_2^2x_3);$$

(v) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1x_2^2x_3 > 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1^2x_2x_3 + 12x_1x_2x_3^2,$ *or equivalently,*

$$(x_1 - x_2 + x_3)^4 > 12(x_1^2x_3^2 + x_2^2x_3^2);$$

(vi) $x_1^4 + x_2^4 + x_3^4 + 6x_2^2x_3^2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1^2x_2x_3 > 6x_1^2x_3^2 + 6x_1^2x_2^2 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2,$ *or equivalently,*

$$(x_2 - x_1 + x_3)^4 > 12(x_1^2x_3^2 + x_2^2x_1^2);$$

(vii) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_3^2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1x_2x_3^2 > 6x_1^2x_2^2 + 6x_2^2x_3^2 + 12x_1x_2^2x_3 + 12x_1^2x_2x_3,$ *or equivalently,*

$$(x_1 + x_2 - x_3)^4 > 12(x_1^2x_2^2 + x_2^2x_3^2).$$

It follows from Theorems 3.8 that the following inequalities are obtained easily.

Theorem 3.13. Let x_1, x_2, x_3 be three nonnegative real numbers. Then

(a) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 + 4x_1^3x_3 \geq 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1^2x_2x_3,$ or equivalently,

$$(x_1 - x_2 + x_3)^4 \geq 8x_1x_3^2(x_3 - 3x_2)$$

with equality if and only if $x_1 = x_2$, $x_3 = 0$ or $x_1 = 0$, $x_2 = x_3$.

(b) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1x_2^2x_3 + 12x_1^2x_2x_3 + 4x_2x_3^3 \geq 4x_1^3x_3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1x_2x_3^2$, or equivalently,

$$(x_2 + x_3 - x_1)^4 \geq 8x_2^2x_3(x_2 - 3x_1)$$

with equality if and only if $x_1 = x_2, x_3 = 0$ or $x_2 = 0, x_1 = x_3$.

(c) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1^2x_2x_3 + 12x_1x_2x_3^2 + 4x_1x_2^3 \geq 4x_1^3x_3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 12x_1x_2^2x_3$, or equivalently,

$$(x_1 + x_2 - x_3)^4 \geq 8x_1^2x_2(x_1 - 3x_3)$$

with equality if and only if $x_1 = 0, x_3 = x_2$ or $x_2 = 0, x_1 = x_3$.

(d) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 12x_1^2x_2x_3 + 12x_1x_2x_3^2 + 12x_1x_2^2x_3 \geq 4x_1^3x_3 + 4x_1x_2^3 + 4x_2^3x_3 + 4x_2x_3^3 + 4x_1^3x_2 + 4x_1x_3^3$, or equivalently,

$$(x_1 + x_2 + x_3)^4 \geq 8(x_1^3x_3 + x_1x_2^3 + x_2^3x_3 + x_2x_3^3 + x_1^3x_2 + x_1x_3^3)$$

with equality if and only if two of x_1, x_2, x_3 are equal and the third is 0.

Proof. It follows from Theorems 3.8 (1) and (2) that the inequalities hold easily. Now we show the equality of (a), and the arguments of (b), (c) and (d) are the same.

$$\begin{aligned} f(x_1, x_2, x_3) =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 - 4x_1^3x_2 + 4x_1^3x_3 - 4x_1x_2^3 - 4x_1x_3^3 \\ & - 4x_2^3x_3 - 4x_2x_3^3 - 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 \\ =& (x_1 - x_2 + x_3)^4 - 8x_1x_3^2(x_3 - 3x_2). \end{aligned}$$

From the arguments of Theorems 3.7 (1), it follows that the zero of function $f(x_1, x_2, x_3)$ must be on the boundary of the non-negative orthant (that is, three coordinate planes), and so, $f(x_1, x_2, x_3) = 0$ if and only if $x_1 = x_2, x_3 = 0$ or $x_1 = 0, x_2 = x_3$.

Similarly, from the arguments of Theorems 3.7 (2), we have the following conclusion also.

Theorem 3.14. Let x_1, x_2, x_3 be three nonnegative real numbers. Then

(e) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 4x_1^3x_2 + 4x_2^3x_3 + 12x_1x_2x_3^2 \geq 4x_1x_2^3 + 4x_1x_3^3 + 4x_2x_3^3 + 4x_1^3x_3 + 12x_1^2x_2x_3 + 12x_1x_2^2x_3$, or equivalently,

$$(x_1 + x_2 - x_3)^4 \geq 8(x_1x_2^3 - x_2^3x_3),$$

with equality if and only if $x_2 = 0$ and $x_1 = x_3$;

(f) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 4x_1x_2^3 + 4x_2^3x_3 + 12x_1^2x_2x_3 \geq 4x_1^3x_2 + 4x_2x_3^3 + 4x_1x_3^3 + 4x_1^3x_3 + 12x_1x_2x_3^2 + 12x_1x_2^2x_3$, or equivalently,

$$(x_3 + x_2 - x_1)^4 \geq 8(x_2x_3^3 - x_1x_2^3),$$

with equality if and only if $x_2 = 0$ and $x_1 = x_3$;

(g) $x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 4x_2x_3^3 + 4x_1^3x_3 + 12x_1x_2^2x_3 \geq 4x_1x_2^3 + 4x_1x_3^3 + 4x_1^3x_2 + 4x_2^3x_3 + 12x_1^2x_2x_3 + 12x_1x_2x_3^2$, or equivalently,

$$(x_1 + x_3 - x_2)^4 \geq 8(x_1x_3^3 - x_2x_3^3),$$

with equality if and only if $x_3 = 0$ and $x_1 = x_2$.

4. Conclusions

For a 4th-order 3-dimensional symmetric tensor with its entries 1 or -1, the analytic necessary and sufficient conditions are established for strict copositivity and copositivity, respectively. These conditions can be applied to verify (strict) copositivity of a general 4th order 3-dimensional symmetric tensor. Several (strict) inequalities of ternary quartic homogeneous polynomial are built by means of these analytic conditions.

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