Characterizations of compactness of fuzzy set space with endograph metric ☆

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Abstract

In this paper, we present the characterizations of total boundedness, relative compactness and compactness in fuzzy set spaces equipped with the endograph metric. The conclusions in this paper significantly improve the corresponding conclusions given in our previous paper [H. Huang, Characterizations of endograph metric and Γ-convergence on fuzzy sets, Fuzzy Sets and Systems 350 (2018), 55-84]. The results in this paper are applicable to fuzzy sets in a general metric space. The results in our previous paper are applicable to fuzzy sets in the m-dimensional Euclidean space $\mathbb{R}^m$, which is a special type of metric space. Furthermore, based on the above results, we give the characterizations of relative compactness, total boundedness and compactness in a kind of common subspaces of general fuzzy sets according to the endograph metric. As an application, we investigate some relationship between the endograph metric and the Γ-convergence on fuzzy sets. This paper is also submitted to arXiv.

Keywords: Compactness; Endograph metric; Γ-convergence; Hausdorff metric

1. Introduction

Fuzzy set is a fundamental tool to investigate fuzzy phenomenon [1–7]. A fuzzy set can be identified with its endograph. The endograph metric H_{end}

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on fuzzy sets is the Hausdorff metric defined on their endographs. It's shown that the endograph metric on fuzzy sets has significant advantages [8–11].

Compactness is one of the central concepts in topology and analysis and is useful in applications (see [6, 12]). The characterizations of compactness in various fuzzy set spaces endowed with different topologies have attracted much attention [13–19].

In [18], we have given the characterizations of total boundedness, relative compactness and compactness of fuzzy set spaces equipped with the endograph metric $H_{\rm end}$.

The results in [18] are applicable to fuzzy sets in the m-dimensional Euclidean space $\mathbb{R}^m$. $\mathbb{R}^m$ is a special type of metric space. In theoretical research and practical applications, fuzzy sets in a general metric space are often used [1, 2, 14, 15].

In this paper, we present the characterizations of total boundedness, relative compactness and compactness of the space fuzzy sets in a general metric space equipped with the endograph metric $H_{\rm end}$. We point out that the characterizations of total boundedness, relative compactness and compactness given in [18] are corollaries of the corresponding characterizations given in this paper.

Furthermore, we discuss the properties of the endograph metric $H_{\rm end}$, and then use these properties and the above characterizations for general fuzzy sets to give the characterizations of relative compactness, total boundedness and compactness in a kind of common subspaces of general fuzzy sets according to the endograph metric $H_{\rm end}$.

As an application of the characterizations of compactness given in this paper, we discuss the relationship between $H_{\rm end}$ metric and Γ-convergence on fuzzy sets.

The remainder of this paper is organized as follows. In Section 2, we recall and give some basic notions and fundamental results related to fuzzy sets and the endograph metric and the Γ-convergence on them. In Section 3, we give representation theorems for various kinds of fuzzy sets which are useful in this paper. In Section 4, we give the characterizations of relatively compact sets, totally bounded sets, and compact sets in space of fuzzy sets in a general metric space equipped with the endograph metric, respectively. In Section 5, based on the characterizations of compactness given in Section 4, we give the characterizations of relatively compact sets, totally bounded sets, and compact sets in a kind of common subspaces of the fuzzy set space discussed in Section 4. In Section 6, as an application of the characterizations of compactness given in Section 4, we investigate the relationship between the endograph metric and the Γ-convergence on fuzzy sets. At last, we draw conclusions in Section 7.

2. Fuzzy sets and endograph metric and Γ-convergence on them

In this section, we recall and give some basic notions and fundamental results related to fuzzy sets and the endograph metric and the Γ-convergence on them. Readers can refer to [1–3, 20, 21] for related contents.

Let $\mathbb{N}$ denote the set of natural numbers. Let $\mathbb{R}$ denote the set of real numbers. Let $\mathbb{R}^m$, $m > 1$, denote the set $\{\langle x_1, \ldots, x_m \rangle : x_i \in \mathbb{R},\ i = 1, \ldots, m\}$. In the sequel, $\mathbb{R}$ is also written as $\mathbb{R}^1$.

Throughout this paper, we suppose that X *is a nonempty set and* d *is the metric on* X. For simplicity, we also use X to denote the metric space (X, d).

The metric $\overline{d}$ on $X \times [0, 1]$ is defined as follows: for $(x, \alpha), (y, \beta) \in X \times [0, 1]$,

$$\overline{d}((x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.$$

Throughout this paper, we suppose that *the metric on* $X \times [0, 1]$ *is* $\overline{d}$. For simplicity, we also use $X \times [0, 1]$ to denote the metric space $(X \times [0, 1], \overline{d})$.

Let $m \in \mathbb{N}$. For simplicity, $\mathbb{R}^m$ is also used to denote the m-dimensional Euclidean space; d_m is used to denote the Euclidean metric on $\mathbb{R}^m$; $\mathbb{R}^m \times [0, 1]$ is also used to denote the metric space $(\mathbb{R}^m \times [0, 1], \overline{d_m})$.

A fuzzy set u in X can be seen as a function $u : X \to [0, 1]$. A subset S of X can be seen as a fuzzy set in X. If there is no confusion, the fuzzy set corresponding to S is often denoted by χ_S; that is,

$$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For simplicity, for $x \in X$, we will use $\widehat{x}$ to denote the fuzzy set $\chi_{\{x\}}$ in X. In this paper, if we want to emphasize a specific metric space X, we will write the fuzzy set corresponding to S in X as $S_{F(X)}$, and the fuzzy set corresponding to $\{x\}$ in X as $\widehat{x}_{F(X)}$.

The symbol $F(X)$ is used to denote the set of all fuzzy sets in X. For $u \in F(X)$ and $\alpha \in [0, 1]$, let $\{u > \alpha\}$ denote the set $\{x \in X : u(x) > \alpha\}$, and let $[u]_\alpha$ denote the α-*cut* of u, i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the topological closure of S in (X, d).

The symbol $K(X)$ and $C(X)$ are used to denote the set of all nonempty compact subsets of X and the set of all nonempty closed subsets of X, respectively. $P(X)$ is used to denote the power set of X, which is the set of all subsets of X.

Let $F_{USC}(X)$ denote the set of all upper semi-continuous fuzzy sets $u : X \to [0, 1]$, i.e.,

$$F_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \cup \{\emptyset\} \text{ for all } \alpha \in [0, 1]\}.$$

Define

$$F_{USCB}(X) := \{u \in F_{USC}(X) : [u]_0 \in K(X) \cup \{\emptyset\}\},$$
$$F_{USCG}(X) := \{u \in F_{USC}(X) : [u]_\alpha \in K(X) \cup \{\emptyset\} \text{ for all } \alpha \in (0, 1]\}.$$

Clearly,

$$F_{USCB}(X) \subseteq F_{USCG}(X) \subseteq F_{USC}(X).$$

Define

$$F_{CON}(X) := \{u \in F(X) : \text{for all } \alpha \in (0, 1],\ [u]_\alpha \text{ is connected in } X\},$$
$$F_{USCCON}(X) := F_{USC}(X) \cap F_{CON}(X),$$
$$F_{USCGCON}(X) := F_{USCG}(X) \cap F_{CON}(X).$$

Let $u \in F_{CON}(X)$. Then $[u]_0 = \overline{\cup_{\alpha>0}[u]_\alpha}$ is connected in X. The proof is as follows.

If $u = \chi_\emptyset$, then $[u]_0 = \emptyset$ is connected in X. If $u \neq \chi_\emptyset$, then there is an $\alpha \in (0, 1]$ such that $[u]_\alpha \neq \emptyset$. Note that $\underline{[u]_\beta \supseteq [u]_\alpha}$ when $\beta \in [0, \alpha]$. Hence $\cup_{0<\beta<\alpha}[u]_\beta$ is connected, and thus $[u]_0 = \overline{\cup_{0<\beta<\alpha}[u]_\beta}$ is connected.

So

$$F_{CON}(X) = \{u \in F(X) : \text{for all } \alpha \in [0, 1],\ [u]_\alpha \text{ is connected in } X\}.$$

Let $F^1_{USC}(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0, 1]$, i.e.,

$$F^1_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}.$$

We introduce some subclasses of $F^1_{USC}(X)$, which will be discussed in this paper. Define

$$F^1_{USCB}(X) := F^1_{USC}(X) \cap F_{USCB}(X),$$

$$F^1_{USCG}(X) := F^1_{USC}(X) \cap F_{USCG}(X),$$
$$F^1_{USCCON}(X) := F^1_{USC}(X) \cap F_{CON}(X),$$
$$F^1_{USCGCON}(X) := F^1_{USCG}(X) \cap F_{CON}(X).$$

Clearly,

$$F^1_{USCB}(X) \subseteq F^1_{USCG}(X) \subseteq F^1_{USC}(X),$$
$$F^1_{USCGCON}(X) \subseteq F^1_{USCCON}(X).$$

Let (X, d) be a metric space. We use $\boldsymbol{H}$ to denote the ***Hausdorff distance*** on $C(X)$ induced by d, i.e.,

$$\boldsymbol{H(U, V)} = \max\{H^*(U, V),\ H^*(V, U)\} \tag{1}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U, V) = \sup_{u \in U} d\,(u, V) = \sup_{u \in U} \inf_{v \in V} d\,(u, v).$$

If there is no confusion, we also use H to denote the Hausdorff distance on $C(X \times [0, 1])$ induced by $\overline{d}$.

The Hausdorff distance on $C(X)$ can be extended to $C(X) \cup \{\emptyset\}$ as follows:

$$H(M_1, M_2) = \begin{cases} H(M_1, M_2), & \text{if } M_1, M_2 \in C(X), \\ +\infty, & \text{if } M_1 = \emptyset \text{ and } M_2 \in C(X), \\ 0, & \text{if } M_1 = M_2 = \emptyset. \end{cases}$$

Remark 2.1. ρ is said to be a *metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be a metric space.

ρ is said to be an *extended metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be an extended metric space.

We can see that for arbitrary metric space (X, d), the Hausdorff distance H on $K(X)$ induced by d is a metric. So the Hausdorff distance H on $K(X \times [0, 1])$ induced by $\overline{d}$ on $X \times [0, 1]$ is a metric.

The Hausdorff distance H on $C(X)$ induced by d on X is an extended metric, but probably not a metric, because $H(A, B)$ could be equal to $+\infty$ for certain metric space X and $A, B \in C(X)$. The Hausdorff distance H on $C(X) \cup \{\emptyset\}$ is an extended metric, but not a metric.

Clearly, if H on $C(X)$ induced by d is not a metric, then H on $C(X\times[0,1])$ induced by $\overline{d}$ is also not a metric. So the Hausdorff distance H on $C(X\times[0,1])$ induced by $\overline{d}$ on $X\times[0,1]$ is an extended metric but probably not a metric.

We can see that H on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and then the same is H on $C(\mathbb{R}^m\times[0,1])$.

In the cases that the Hausdorff distance H is a metric, we call the Hausdorff distance the Hausdorff metric. In the cases that the Hausdorff distance H is an extended metric, we call the Hausdorff distance the Hausdorff extended metric. In this paper, for simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric.

For $u\in F(X)$, define

$$\begin{aligned}\operatorname{end} u &:= \{(x,t)\in X\times[0,1] : u(x)\geq t\},\\ \operatorname{send} u &:= \{(x,t)\in X\times[0,1] : u(x)\geq t\}\cap([u]_0\times[0,1]).\end{aligned}$$

$\operatorname{end} u$ and $\operatorname{send} u$ are called the endograph and the sendograph of u, respectively.

Let $u\in F(X)$. The following properties (i)-(iii) are equivalent:
(i) $u\in F_{USC}(X)$;
(ii) $\operatorname{end} u$ is closed in $(X\times[0,1],\overline{d})$;
(iii) $\operatorname{send} u$ is closed in $(X\times[0,1],\overline{d})$.

(i)⇒(ii). Assume that (i) is true. To show that (ii) is true, let $\{(x_n,\alpha_n)\}$ be a sequence in $\operatorname{end} u$ which converges to (x,α) in $X\times[0,1]$, we only need to show that $(x,\alpha)\in\operatorname{end} u$. Since u is upper semi-continuous, then $u(x)\geq\limsup_{n\to\infty}u(x_n)\geq\lim_{n\to\infty}\alpha_n=\alpha$. Thus $(x,\alpha)\in\operatorname{end} u$. So (ii) is true.

(ii)⇒(iii). Assume that (ii) is true. Note that $[u]_0\times[0,1]$ is closed in $X\times[0,1]$, then $\operatorname{send} u=\operatorname{end} u\cap([u]_0\times[0,1])$ is closed in $X\times[0,1]$. So (iii) is true.

(iii)⇒(i). Assume that (iii) is true. To show that (i) is true, let $\alpha\in[0,1]$ and suppose that $\{x_n\}$ is a sequence in $[u]_\alpha$ which converges to x in X, we only need to show that $x\in[u]_\alpha$. Note that $\{(x_n,\alpha)\}$ converges to (x,α) in $X\times[0,1]$, and that the sequence $\{(x_n,\alpha)\}$ is in $\operatorname{send} u$. Hence from the closedness of $\operatorname{send} u$, it follows that $(x,\alpha)\in\operatorname{send} u$, which means that $x\in[u]_\alpha$. So (i) is true.

Let $u\in F(X)$. Clearly $X\times\{0\}\subseteq\operatorname{end} u$. So $\operatorname{end} u\neq\emptyset$. We can see that $\operatorname{send} u=\emptyset$ if and only if $u=\emptyset_{F(X)}$.

From the above discussions, we know that $u\in F_{USC}(X)$ if and only if $\operatorname{end} u\in C(X\times[0,1])$.

Kloeden [8] introduced the ***endograph metric*** $\boldsymbol{H_{\text{end}}}$. For $u, v \in F_{USC}(X)$,

$$\boldsymbol{H_{\text{end}}(u, v)} := H(\text{end}\, u, \text{end}\, v),$$

where H is the Hausdorff metric on $C(X \times [0, 1])$ induced by $\overline{d}$ on $X \times [0, 1]$.

Rojas-Medar and Román-Flores [20] introduced the $\boldsymbol{\Gamma}$***-convergence*** of a sequence of upper semi-continuous fuzzy sets based on the Kuratowski convergence of a sequence of sets in a metric space.

Let (X, d) be a metric space. Let C be a set in X and $\{C_n\}$ a sequence of sets in X. $\{C_n\}$ is said to ***Kuratowski converge*** to C according to (X, d), if

$$C = \liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n,$$

where

$$\liminf_{n\to\infty} C_n = \{x \in X : \ x = \lim_{n\to\infty} x_n, x_n \in C_n\},$$

$$\limsup_{n\to\infty} C_n = \{x \in X : \ x = \lim_{j\to\infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} C_m}.$$

In this case, we'll write $\boldsymbol{C = \lim^{(K)}_{n\to\infty} C_n}$ according to (X, d). If there is no confusion, we will not emphasize the metric space (X, d) and write $\{C_n\}$ ***Kuratowski converges*** to C or $\boldsymbol{C = \lim^{(K)}_{n\to\infty} C_n}$ for simplicity.

Remark 2.2. Theorem 5.2.10 in [22] pointed out that, in a first countable Hausdorff topological space, a sequence of sets is Kuratowski convergent is equivalent to this sequence is Fell topology convergent. A metric space is of course a first countable Hausdorff topological space.

Definition 3.1.4 in [23] gives the definitions of $\liminf C_n$, $\limsup C_n$ and $\lim C_n$ for a net of subsets $\{C_n, n \in D\}$ in a topological space. When $\{C_n, n = 1, 2, \ldots\}$ is a sequence of subsets of a metric space, $\liminf C_n$, $\limsup C_n$ and $\lim C_n$ according to Definition 3.1.4 in [23] are $\liminf_{n\to\infty} C_n$, $\limsup_{n\to\infty} C_n$ and $\lim^{(K)}_{n\to\infty} C_n$ according to the above definitions, respectively.

Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F_{USC}(X)$. $\{u_n\}$ is said to $\boldsymbol{\Gamma}$-***converge*** to u, denoted by $\boldsymbol{u = \lim^{(\Gamma)}_{n\to\infty} u_n}$, if $\text{end}\, u = \lim^{(K)}_{n\to\infty} \text{end}\, u_n$ according to $(X \times [0, 1], \overline{d})$.

The following Theorem 2.3 is an already known conclusion, which is useful in this paper.

Theorem 2.3. *Suppose that C, C_n are sets in $C(X)$, $n = 1, 2, \ldots$. Then $H(C_n, C) \to 0$ implies that $\lim^{(K)}_{n\to\infty} C_n = C$.*

Remark 2.4. Theorem 2.3 implies that for a sequence $\{u_n\}$ in $F_{USC}(X)$ and an element u in $F_{USC}(X)$, if $H_{\text{end}}(u_n, u) \to 0$ as $n \to \infty$, then $\lim^{(\Gamma)}_{n\to\infty} u_n = u$. However, the converse is false. See Example 4.1 in [18].

Theorem 2.3 can be shown in a similar fashion to Theorem 4.1 in [18]. In Theorem 2.3, we exclude the case that $C = \emptyset$.

Remark 2.5. Let $\{u_n\}$ be a sequence in $F_{USC}(X)$ and let $\{v_n\}$ be a subsequence of $\{u_n\}$. We can see that

$$\liminf_{n\to\infty} u_n \subseteq \liminf_{n\to\infty} v_n \subseteq \limsup_{n\to\infty} v_n \subseteq \limsup_{n\to\infty} u_n.$$

So if there is a $u \in F_{USC}(X)$ with $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, then $\lim^{(\Gamma)}_{n\to\infty} v_n = u$.

Clearly, $\lim^{(\Gamma)}_{n\to\infty} v_n = u$ does not necessarily imply that $\lim^{(\Gamma)}_{n\to\infty} u_n = u$. A simple example is given below.

For $n = 1, 2, \ldots$, let $v_n = \widehat{1}_{F(\mathbb{R})}$. For $n = 1, 2, \ldots$, let $u_n \in F_{USC}(\mathbb{R})$ defined by

$$u_n = \begin{cases} \widehat{1}_{F(\mathbb{R})}, & n \text{ is odd}, \\ \widehat{3}_{F(\mathbb{R})}, & n \text{ is even}. \end{cases}$$

Then $\{v_n\}$ is a subsequence of $\{u_n\}$. We can see that $\lim^{(\Gamma)}_{n\to\infty} v_n = \widehat{1}_{F(\mathbb{R})}$. However $\lim^{(\Gamma)}_{n\to\infty} u_n$ does not exist because

$$\liminf_{n\to\infty} u_n = \mathbb{R} \times \{0\} \subsetneqq \operatorname{end} \widehat{1}_{F(\mathbb{R})} \vee \operatorname{end} \widehat{3}_{F(\mathbb{R})} = \limsup_{n\to\infty} u_n.$$

In this paper, for a metric space (Y, ρ) and a subset S in Y, we still use ρ to denote the induced metric on S by ρ.

3. Representation theorems for various kinds of fuzzy sets

In this section, we give representation theorems for various kinds of fuzzy sets. These representation theorems are useful in this paper.

The following representation theorem should be a known conclusion. In this paper we assume that $\sup \emptyset = 0$.

Theorem 3.1. *Let Y be a nonempty set. If $u \in F(Y)$, then for all $\alpha \in (0, 1]$, $[u]_\alpha = \cap_{\beta<\alpha} [u]_\beta$.*

Conversely, suppose that $\{v_\alpha : \alpha \in (0,1]\}$ is a family of sets in Y with $v_\alpha = \cap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$. Define $u \in F(Y)$ by

$$u(x) := \sup\{\alpha : x \in v_\alpha\}$$

for each $x \in Y$. Then u is the unique fuzzy set in Y satisfying that $[u]_\alpha = v_\alpha$ for all $\alpha \in (0,1]$.

Proof. Let $u \in F(Y)$ and $\alpha \in (0,1]$. For each $x \in Y$, $x \in [u]_\alpha \Leftrightarrow u(x) \geq \alpha \Leftrightarrow$ for each $\beta < \alpha$, $u(x) \geq \beta \Leftrightarrow$ for each $\beta < \alpha$, $x \in [u]_\beta$. So $[u]_\alpha = \cap_{\beta<\alpha}[u]_\beta$.

Conversely, suppose that $\{v_\alpha : \alpha \in (0,1]\}$ is a family of sets in Y with $v_\alpha = \cap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0,1]$. Let $u \in F(Y)$ defined by

$$u(x) := \sup\{\alpha : x \in v_\alpha\}$$

for each $x \in Y$. Firstly, we show that for each $\alpha \in (0,1]$, $[u]_\alpha = v_\alpha$. To do this, let $\alpha \in (0,1]$. We only need to verify that $[u]_\alpha \supseteq v_\alpha$ and $[u]_\alpha \subseteq v_\alpha$.

Let $x \in v_\alpha$. Then clearly $u(x) \geq \alpha$, i.e. $x \in [u]_\alpha$. So $[u]_\alpha \supseteq v_\alpha$.

Let $x \in [u]_\alpha$. Then $\sup\{\beta : x \in v_\beta\} = u(x) \geq \alpha$. Hence there exists a sequence $\{\beta_n,\ n = 1, 2, \ldots\}$ such that $1 \geq \beta_n \geq \alpha - 1/n$ and $x \in v_{\beta_n}$. Set $\gamma = \sup_{n=1}^{+\infty} \beta_n$. Then $1 \geq \gamma \geq \alpha$ and thus $x \in \cap_{n=1}^{+\infty} v_{\beta_n} = v_\gamma \subseteq v_\alpha$. So $[u]_\alpha \subseteq v_\alpha$.

Now we show the uniqueness of u. To do this, assume that v is a fuzzy set in Y satisfying that $[v]_\alpha = v_\alpha$ for all $\alpha \in (0,1]$. Then for each $x \in Y$,

$$v(x) = \sup\{\alpha : x \in [v]_\alpha\} = \sup\{\alpha : x \in v_\alpha\} = u(x).$$

So $u = v$.

$\square$

Remark 3.2. We can't find the original reference which gave Theorem 3.1, so we give a proof here for the self-containing of this paper. Theorem 3.1 and its proof are essentially the same as the Theorem 7.10 in P27 of chinaXiv:202110.00083v4 and its proof since the uniqueness of u is obvious.

From Theorem 3.1, it follows immediately below representation theorems for $F_{USC}(X)$, $F^1_{USC}(X)$, $F_{USCG}(X)$, $F_{CON}(X)$, $F_{USCB}(X)$, and $F^1_{USCB}(X)$.

Proposition 3.3. *Let (X, d) be a metric space. If $u \in F_{USC}(X)$ (respectively, $u \in F^1_{USC}(X)$, $u \in F_{USCG}(X)$, $u \in F_{CON}(X)$), then*
(i) $[u]_\alpha \in C(X) \cup \{\emptyset\}$ (respectively, $[u]_\alpha \in C(X)$, $[u]_\alpha \in K(X) \cup \{\emptyset\}$, $[u]_\alpha$ is connected in (X, d)) for all $\alpha \in (0, 1]$, and
(ii) $[u]_\alpha = \bigcap_{\beta<\alpha}[u]_\beta$ for all $\alpha \in (0, 1]$.

Conversely, suppose that the family of sets $\{v_\alpha : \alpha \in (0, 1]\}$ satisfies conditions (i) and (ii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then u is the unique fuzzy set in X satisfying that $[u]_\alpha = v_\alpha$ for each $\alpha \in (0, 1]$. Moreover, $u \in F_{USC}(X)$ (respectively, $u \in F^1_{USC}(X)$, $u \in F_{USCG}(X)$, $u \in F_{CON}(X)$).

Proof. The proof is routine. We only show the case of $F_{USC}(X)$. The other cases can be verified similarly.

If $x \in F_{USC}(X)$, then clearly (i) is true. From Theorem 3.1, (ii) is true.

Conversely, suppose that the family of sets $\{v_\alpha : \alpha \in (0, 1]\}$ satisfies conditions (i) and (ii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then by Theorem 3.1, u is the unique fuzzy set in X satisfying that $[u]_\alpha = v_\alpha$ for each $\alpha \in (0, 1]$. Since $\{[u]_\alpha, \alpha \in (0, 1]\}$ satisfies condition (i), $u \in F_{USC}(X)$.

□

Proposition 3.4. *Let (X, d) be a metric space. If $u \in F_{USCB}(X)$ (respectively, $u \in F^1_{USCB}(X)$), then*
(i) $[u]_\alpha \in K(X) \cup \{\emptyset\}$ (respectively, $[u]_\alpha \in K(X)$) for all $\alpha \in [0, 1]$,
(ii) $[u]_\alpha = \bigcap_{\beta<\alpha}[u]_\beta$ for all $\alpha \in (0, 1]$, and
(iii) $[u]_0 = \overline{\bigcup_{\beta>0}[u]_\beta}$.

Conversely, suppose that the family of sets $\{v_\alpha : \alpha \in [0, 1]\}$ satisfies conditions (i) through (iii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then u is the unique fuzzy set in X satisfying that $[u]_\alpha = v_\alpha$ for each $\alpha \in [0, 1]$. Moreover, $u \in F_{USCB}(X)$ (respectively, $u \in F^1_{USCB}(X)$).

Proof. The proof is routine. We only show the case of $F_{USCB}(X)$. The case of $F^1_{USCB}(X)$ can be verified similarly.

If $x \in F_{USCB}(X)$, then clearly (i) is true. By Theorem 3.1, (ii) is true. From the definition of $[u]_0$, (iii) is true.

Conversely, suppose that the family of sets $\{v_\alpha : \alpha \in [0, 1]\}$ satisfies conditions (i) through (iii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then by Theorem 3.1, u is the unique fuzzy set in X satisfying that $[u]_\alpha = v_\alpha$ for each $\alpha \in (0,1]$. Clearly $[u]_0 = \overline{\bigcup_{\beta>0}[u]_\beta} = \overline{\bigcup_{\beta>0} v_\beta} = v_0$. Since $\{[u]_\alpha, \alpha \in [0,1]\}$ satisfies condition (i), $u \in F_{USCB}(X)$.

$\square$

Similarly, we can obtain the representation theorems for $F_{USCCON}(X)$, $F_{USCGCON}(X)$, $F^1_{USCCON}(X)$, etc.

Based on these representation theorems, we can define a fuzzy set or a certain type fuzzy set by giving the family of its α-cuts. In the sequel, we will directly point out that what we defined is a fuzzy set or a certain type fuzzy set without saying which representation theorem is used since it is easy to see.

4. Characterization of compactness in $(F_{USCG}(X), H_{\text{end}})$

In this section, we give the characterizations of relatively compact sets, totally bounded sets, and compact sets in $(F_{USCG}(X), H_{\text{end}})$, respectively. We point out that these results improve the characterizations of relatively compact sets, totally bounded sets, and compact sets in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ given in our previous work [18], respectively.

- A subset Y of a topological space Z is said to be *compact* if for every set I and every family of open sets, O_i, $i \in I$, such that $Y \subset \bigcup_{i\in I} O_i$ there exists a finite family $O_{i_1}, O_{i_2} \ldots, O_{i_n}$ such that $Y \subseteq O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_n}$. In the case of a metric topology, the criterion for compactness becomes that any sequence in Y has a subsequence convergent in Y.

- A *relatively compact* subset Y of a topological space Z is a subset with compact closure. In the case of a metric topology, the criterion for relative compactness becomes that any sequence in Y has a subsequence convergent in X.

- Let (X, d) be a metric space. A set U in X is *totally bounded* if and only if, for each $\varepsilon > 0$, it contains a finite ε approximation, where an ε approximation to U is a subset S of U such that $d(x, S) < \varepsilon$ for each $x \in U$. An ε approximation to U is also called an ε-net of U.

Let (X, d) be a metric space. A set U is compact in (X, d) implies that U is relatively compact in (X, d), which in turn implies that U is totally bounded in (X, d). Let Y be a subset of X and A a subset of Y. Then A is totally bounded in (Y, d) if and only if A is totally bounded in (X, d).

Theorem 4.1. *[19] Let* (X, d) *be a metric space and* $\mathcal{D} \subseteq K(X)$*. Then* $\mathcal{D}$ *is totally bounded in* $(K(X), H)$ *if and only if* $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ *is totally bounded in* (X, d)*.*

Theorem 4.2. *[14] Let* (X, d) *be a metric space and* $\mathcal{D} \subseteq K(X)$*. Then* $\mathcal{D}$ *is relatively compact in* $(K(X), H)$ *if and only if* $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ *is relatively compact in* (X, d)*.*

Theorem 4.3. *[19] Let* (X, d) *be a metric space and* $\mathcal{D} \subseteq K(X)$*. Then the following are equivalent:*
(i) $\mathcal{D}$ *is compact in* $(K(X), H)$*;*
(ii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ *is relatively compact in* (X, d) *and* $\mathcal{D}$ *is closed in* $(K(X), H)$*;*
(iii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ *is compact in* (X, d) *and* $\mathcal{D}$ *is closed in* $(K(X), H)$*.*

Let $u \in F_{USC}(X)$. Define $\boldsymbol{u^e} = \operatorname{end} u$. Let A be a subset of $F_{USC}(X)$. Define $\boldsymbol{A^e} = \{u^e : u \in A\}$. Clearly $F_{USC}(X)^e \subseteq C(X \times [0, 1])$.

Define $g : (F_{USC}(X), H_{\text{end}}) \to (C(X \times [0, 1]), H)$ given by $g(u) = \operatorname{end} u$. Then

- g is an isometric embedding of $(F_{USC}(X), H_{\text{end}})$ in $(C(X \times [0, 1]), H)$,
- $g(F_{USC}(X)) = F_{USC}(X)^e$, and
- $(F_{USC}(X), H_{\text{end}})$ is isometric to $(F_{USC}(X)^e, H)$.

The following representation theorem for $F_{USC}(X)^e$ follows immediately from Proposition 3.3.

Proposition 4.4. *Let* U *be a subset of* $X \times [0, 1]$*. Then* $U \in F_{USC}(X)^e$ *if and only if the following properties (i)-(iii) are true.*
(i) For each $\alpha \in (0, 1]$*,* $\langle U \rangle_\alpha \in C(X) \cup \{\emptyset\}$*.*
(ii) For each $\alpha \in (0, 1]$*,* $\langle U \rangle_\alpha = \bigcap_{\beta<\alpha} \langle U \rangle_\beta$*.*
(iii) $\langle U \rangle_0 = X$*.*

Proposition 4.5. *Let* $U \in C(X \times [0, 1])$*. Then the following (i) and (ii) are equivalent.*
(i) For each α *with* $0 < \alpha \le 1$*,* $\langle U \rangle_\alpha = \bigcap_{\beta<\alpha} \langle U \rangle_\beta$*.*
(ii) For each α, β *with* $0 \le \beta < \alpha \le 1$*,* $\langle U \rangle_\alpha \subseteq \langle U \rangle_\beta$*.*

Proof. The proof is routine. (i)$\Rightarrow$(ii) is obviously.

Suppose that (ii) is true. To show that (i) is true, let $\alpha \in (0,1]$. From (ii), $\langle U\rangle_\alpha \subseteq \bigcap_{\beta<\alpha}\langle U\rangle_\beta$. So we only need to prove that $\langle U\rangle_\alpha \supseteq \bigcap_{\beta<\alpha}\langle U\rangle_\beta$. To do this, let $x \in \bigcap_{\beta<\alpha}\langle U\rangle_\beta$. This means that $(x,\beta) \in U$ for $\beta \in [0,\alpha)$. Since $\lim_{\beta\to\alpha-} \overline{d}((x,\beta),(x,\alpha)) = 0$, from the closedness of U, it follows that $(x,\alpha) \in U$. Hence $x \in \langle U\rangle_\alpha$. Thus $\langle U\rangle_\alpha \supseteq \bigcap_{\beta<\alpha}\langle U\rangle_\beta$ from the arbitrariness of x in $\bigcap_{\beta<\alpha}\langle U\rangle_\beta$. So (ii)$\Rightarrow$(i).

$\square$

Proposition 4.6. *Let $U \in C(X\times[0,1])$. Then $U \in F_{USC}(X)^e$ if and only if U has the following properties:*
(i) for each α,β with $0 \le \beta < \alpha \le 1$, $\langle U\rangle_\alpha \subseteq \langle U\rangle_\beta$, and
(ii) $\langle U\rangle_0 = X$.

Proof. Since $U \in C(X\times[0,1])$, then clearly $\langle U\rangle_\alpha \in C(X)\cup\{\emptyset\}$ for all $\alpha \in [0,1]$. Thus the desired result follows immediately from Propositions 4.4 and 4.5.

$\square$

As a shorthand, we denote the sequence $x_1, x_2, \ldots, x_n, \ldots$ by $\{x_n\}$.

Proposition 4.7. *$F_{USC}(X)^e$ is a closed subset of $(C(X\times[0,1]), H)$.*

Proof. Let $\{u_n^e : n = 1,2,\ldots\}$ be a sequence in $F_{USC}(X)^e$ with $\{u_n^e\}$ converging to U in $(C(X\times[0,1]), H)$. To show the desired result, we only need to show that $U \in F_{USC}(X)^e$.

We claim that
(i) for each α,β with $0 \le \beta < \alpha \le 1$, $\langle U\rangle_\alpha \subseteq \langle U\rangle_\beta$;
(ii) $\langle U\rangle_0 = X$.

To show (i), let α,β in $[0,1]$ with $\beta < \alpha$, and let $x \in \langle U\rangle_\alpha$, i.e. $(x,\alpha) \in U$. By Theorem 2.3, $\lim^{(K)}_{n\to\infty} u_n^e = U$. Then there is a sequence $\{(x_n,\alpha_n)\}$ satisfying $(x_n,\alpha_n) \in u_n^e$ for $n = 1,2,\ldots$ and $\lim_{n\to\infty} \overline{d}((x_n,\alpha_n),(x,\alpha)) = 0$. Hence there is an N such that $\alpha_n > \beta$ for all $n \ge N$. Thus $(x_n,\beta) \in u_n^e$ for $n \ge N$. Note that $\lim_{n\to\infty} \overline{d}((x_n,\beta),(x,\beta)) = 0$. Then $(x,\beta) \in \lim^{(K)}_{n\to\infty} u_n^e = U$. This means that $x \in \langle U\rangle_\beta$. So (i) is true.

Clearly $\langle U\rangle_0 \subseteq X$. From $\lim^{(K)}_{n\to\infty} u_n^e = U$ and $\langle u_n^e\rangle_0 = X$, we have that $\langle U\rangle_0 \supseteq X$. Thus $\langle U\rangle_0 = X$. So (ii) is true.

By Proposition 4.6, (i) and (ii) imply that $U \in F_{USC}(X)^e$.

$\square$

Remark 4.8. Let $a \in [0,1]$. From Proposition 5.1, we can deduce that $F'^{a}_{USC}(X)$ is a closed subset of $(F_{USC}(X), H_{\rm end})$. Then by Proposition 4.7, we have that $F'^{a}_{USC}(X)^e$ is a closed subset of $(C(X \times [0,1]), H)$.

We use $(\widetilde{X}, \widetilde{d})$ to denote the completion of (X, d). We see (X, d) as a subspace of $(\widetilde{X}, \widetilde{d})$.

If there is no confusion, we also use H to denote the Hausdorff metric on $C(\widetilde{X})$ induced by $\widetilde{d}$. We also use H to denote the Hausdorff metric on $C(\widetilde{X} \times [0,1])$ induced by $\overline{\widetilde{d}}$. We also use $H_{\rm end}$ to denote the endograph metric on $F_{USC}(\widetilde{X})$ given by using H on $C(\widetilde{X} \times [0,1])$.

$F(X)$ can be naturally embedded into $F(\widetilde{X})$. An embedding $\boldsymbol{j}$ from $F(X)$ to $F(\widetilde{X})$ is defined as follows.

Let $u \in F(X)$. We can define $j(u) \in F(\widetilde{X})$ as

$$j(u)(t) = \begin{cases} u(t), & t \in X, \\ 0, & t \in \widetilde{X} \setminus X. \end{cases}$$

Let $U \subseteq X$. If U is compact in (X, d), then U is compact in $(\widetilde{X}, \widetilde{d})$. So if $u \in F_{USCG}(X)$, then $j(u) \in F_{USCG}(\widetilde{X})$ because $[j(u)]_\alpha = [u]_\alpha \subseteq K(\widetilde{X}) \cup \{\emptyset\}$ for each $\alpha \in (0,1]$.

We can see that for $u, v \in F_{USCG}(X)$, $H_{\rm end}(u, v) = H_{\rm end}(j(u), j(v))$. So $j|_{F_{USCG}(X)}$ is an isometric embedding of $(F_{USCG}(X), H_{\rm end})$ in $(F_{USCG}(\widetilde{X}), H_{\rm end})$.

Since $(F_{USCG}(X), H_{\rm end})$ can be embedded isometrically in $(F_{USCG}(\widetilde{X}), H_{\rm end})$, in the sequel, we treat $(F_{USCG}(X), H_{\rm end})$ as a metric subspace of $(F_{USCG}(\widetilde{X}), H_{\rm end})$ by identifying u in $F_{USCG}(X)$ with $j(u)$ in $F_{USCG}(\widetilde{X})$. So a subset U of $F_{USCG}(X)$ can be seen as a subset of $F_{USCG}(\widetilde{X})$.

Suppose that U is a subset of $F_{USC}(X)$ and $\alpha \in [0,1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u \in U} [u]_\alpha$, and
- $U_\alpha := \{[u]_\alpha : u \in U\}$.

Here we mention that an empty union is $\emptyset$.

Lemma 4.9. *Let U be a subset of $F_{USCG}(X)$. If U is totally bounded in $(F_{USCG}(X), H_{\rm end})$, then $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0,1]$.*

Proof. The proof is similar to that of the necessity part of Theorem 7.8 in [19].

Let $\alpha \in (0,1]$. To show that $U(\alpha)$ is totally bounded in X, we only need to show that each sequence in $U(\alpha)$ has a Cauchy subsequence.

Let $\{x_n\}$ be a sequence in $U(\alpha)$. Then there is a sequence $\{u_n\}$ in U with $x_n \in [u_n]_\alpha$ for $n = 1, 2, \ldots$. Since U is totally bounded in $(F_{USCG}(X), H_{\text{end}})$, $\{u_n\}$ has a Cauchy subsequence $\{u_{n_l}\}$ in $(F_{USCG}(X), H_{\text{end}})$. So given $\varepsilon \in (0, \alpha)$, there is a $K(\varepsilon) \in \mathbb{N}$ such that

$$H_{\text{end}}(u_{n_l}, u_{n_K}) < \varepsilon$$

for all $l \geq K$. Thus

$$H^*([u_{n_l}]_\alpha, [u_{n_K}]_{\alpha-\varepsilon}) < \varepsilon \tag{2}$$

for all $l \geq K$. From (2) and the arbitrariness of ε, $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$ is totally bounded in (X, d). Thus $\{x_{n_l}\}$, which is a subsequence of $\{x_n\}$, has a Cauchy subsequence, and so does $\{x_n\}$.

□

Remark 4.10. It is easy to see that for a totally bounded set U in $(F_{USCG}(X), H_{\text{end}})$ and $\alpha \in (0,1]$, $U(\alpha) = \emptyset$ is possible even if $U \neq \emptyset$.

For $D \subseteq X \times [0,1]$ and $\alpha \in [0,1]$, define $\boldsymbol{\langle D \rangle_\alpha} := \{x : (x, \alpha) \in D\}$.

Let $u \in F(X)$ and $0 \leq r \leq t \leq 1$. We use the symbol $\mathbf{end}_{\boldsymbol{r}}^{\boldsymbol{t}}\, \boldsymbol{u}$ to denote the subset of $\text{end}\, u$ given by

$$\text{end}_r^t\, u := \text{end}\, u \cap ([u]_r \times [r, t]).$$

For simplicity, we write $\text{end}_r^1\, u$ as $\text{end}_r\, u$. We can see that $\text{end}_0\, u = \text{send}\, u$.

Theorem 4.11. *Let U be a subset of $F_{USCG}(X)$. Then U is relatively compact in $(F_{USCG}(X), H_{\text{end}})$ if and only if $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0,1]$.*

Proof. ***Necessity***. Suppose that U is relatively compact in $(F_{USCG}(X), H_{\text{end}})$. Let $\alpha \in (0,1]$. Then by Lemma 4.9, $U(\alpha)$ is totally bounded. Hence $U(\alpha)$ is relatively compact in $(\widetilde{X}, \widetilde{d})$.

To show that $U(\alpha)$ is relatively compact in (X, d), we proceed by contradiction. If this were not the case, then there exists a sequence $\{x_n\}$ in $U(\alpha)$ such that $\{x_n\}$ converges to $x \in \widetilde{X} \setminus X$ in $(\widetilde{X}, \widetilde{d})$.

Assume that $x_n \in [u_n]_\alpha$ and $u_n \in U$, $n = 1, 2, \ldots$. From the relative compactness of U, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges to $u \in F_{USCG}(X)$. Since $F_{USCG}(X)$ can be seen as a subspace of $F_{USCG}(\widetilde{X})$, we obtain that $\{u_{n_k}\}$ converges to u in $(F_{USCG}(\widetilde{X}), H_{\rm end})$. By Theorem 2.3, $\lim^{(K)}_{k\to\infty} u^e_{n_k} = u^e$ according to $(\widetilde{X} \times [0,1], \overline{\widetilde{d}})$. Notice that $(x_{n_k}, \alpha) \in u^e_{n_k}$ for $k = 1, 2, \ldots$, and $\{(x_{n_k}, \alpha)\}$ converges to (x, α) in $(\widetilde{X} \times [0,1], \overline{\widetilde{d}})$. Thus $(x, \alpha) \in u^e$, which contradicts $x \in \widetilde{X} \setminus X$.

It can be seen that the necessity part of Theorem 7.10 in [19] can be verified in a similar manner to that in the necessity part of this theorem.

Sufficiency. Suppose that $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0,1]$. To show that U is relatively compact in $(F_{USCG}(X), H_{\rm end})$, we only need to show that each sequence in U has a convergent subsequence in $(F_{USCG}(X), H_{\rm end})$.

Let $\{u_n\}$ be a sequence in U. If $\liminf_{n\to\infty} S_{u_n} = 0$, i.e. there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\lim_{k\to\infty} S_{u_{n_k}} = 0$. Then clearly $H_{\rm end}(u_{n_k}, \emptyset_{F(X)}) = S_{u_{n_k}} \to 0$ as $k \to \infty$. Since $\emptyset_{F(X)} \in F_{USCG}(X)$, $\{u_{n_k}\}$ is a convergent subsequence in $(F_{USCG}(X), H_{\rm end})$.

If $\liminf_{n\to\infty} S_{u_n} > 0$, then there is a $\xi > 0$ and an $N \in \mathbb{N}$ such that $[u_n]_\xi \neq \emptyset$ for all $n \geq N$.

First we claim the following property:

(a) Let $\alpha \in (0,1]$ and S be a subset of U with $[u]_\alpha \neq \emptyset$ for each $u \in S$. Then $\{{\rm end}_\alpha u : u \in S\}$ is a relatively compact set in $(K(X \times [\alpha, 1]), H)$.

It can be seen that for each $u \in S$, ${\rm end}_\alpha\, u \in K(X \times [\alpha, 1])$.

As $U(\alpha)$ is relatively compact in (X, d), $U(\alpha) \times [\alpha, 1]$ is relatively compact in $(X \times [\alpha, 1], \overline{d})$. Since $\bigcup_{u\in S} {\rm end}_\alpha u$ is a subset of $U(\alpha) \times [\alpha, 1]$, then $\bigcup_{u\in S} {\rm end}_\alpha u$ is also a relatively compact set in $(X \times [\alpha, 1], \overline{d})$. Thus by Theorem 4.2, $\{{\rm end}_\alpha u : u \in S\}$ is relatively compact in $(K(X \times [\alpha, 1]), H)$. So affirmation (a) is true.

Take a sequence $\{\alpha_k,\ k = 1, 2, \ldots\}$ which satisfies that $0 < \alpha_{k+1} < \alpha_k \leq \min\{\xi, \frac{1}{k}\}$ for $k = 1, 2, \ldots$. We can see that $\alpha_k \to 0$ as $k \to \infty$.

By affirmation (a), $\{{\rm end}_{\alpha_1} u_n : n = N, N+1, \ldots\}$ is relatively compact in $(K(X \times [\alpha_1, 1]), H)$. So there is a subsequence $\{u^{(1)}_n\}$ of $\{u_n : n \geq N\}$ and $v^1 \in K(X \times [\alpha_1, 1])$ such that $H({\rm end}_{\alpha_1} u^{(1)}_n, v^1) \to 0$. Clearly, $\{u^{(1)}_n\}$ is also a subsequence of $\{u_n\}$.

Again using affirmation (a), $\{\mathrm{end}_{\alpha_2} u_n^{(1)}\}$ is relatively compact in $(K(X \times [\alpha_2, 1]), H)$. So there is a subsequence $\{u_n^{(2)}\}$ of $\{u_n^{(1)}\}$ and $v^2 \in K(X \times [\alpha_2, 1])$ such that $H(\mathrm{end}_{\alpha_2} u_n^{(2)}, v^2) \to 0$.

Repeating the above procedure, we can obtain $\{u_n^{(k)}\}$ and $v^k \in K(X \times [\alpha_k, 1])$, $k = 1, 2, \ldots$, such that for each $k = 1, 2, \ldots$, $\{u_n^{(k+1)}\}$ is a subsequence of $\{u_n^{(k)}\}$ and $H(\mathrm{end}_{\alpha_k} u_n^{(k)}, v^k) \to 0$.

We claim that

(b) Let k_1 and k_2 be in $\mathbb{N}$ with $k_1 < k_2$. Then
(i) $\langle v^{k_1} \rangle_{\alpha_{k_1}} \subseteq \langle v^{k_2} \rangle_{\alpha_{k_1}}$,
(ii) $\langle v^{k_1} \rangle_{\alpha} = \langle v^{k_2} \rangle_{\alpha}$ when $\alpha \in (\alpha_{k_1}, 1]$,
(iii) $v^k \subseteq v^{k+1}$ for $k = 1, 2, \ldots$.

Note that $\{u_n^{(k_2)}\}$ is a subsequence of $\{u_n^{(k_1)}\}$ and that $\alpha_{k_2} < \alpha_{k_1}$. Thus by Theorem 2.3, for each $\alpha \in [\alpha_{k_1}, 1]$,

$$\begin{aligned}
&\langle v^{k_1} \rangle_{\alpha} \times \{\alpha\} \\
&= \liminf_{n\to\infty} \mathrm{end}_{\alpha_{k_1}} u_n^{(k_1)} \cap (X \times \{\alpha\}) \\
&\subseteq \liminf_{n\to\infty} \mathrm{end}_{\alpha_{k_2}} u_n^{(k_2)} \cap (X \times \{\alpha\}) \\
&= \langle v^{k_2} \rangle_{\alpha} \times \{\alpha\}.
\end{aligned} \tag{3}$$

So (i) is true.

Let $\alpha \in [0, 1]$ with $\alpha > \alpha_{k_1}$. Observe that if a sequence $\{(x_m, \beta_m)\}$ converges to a point (x, α) as $m \to \infty$ in $(X \times [0, 1], \overline{d})$, then there is an M such that for all $m > M$, $\beta_m > \alpha_{k_1}$, i.e. $(x_m, \beta_m) \in X \times (\alpha_{k_1}, 1]$. Thus by Theorem 2.3, for each $\alpha \in (\alpha_{k_1}, 1]$,

$$\begin{aligned}
&\langle v^{k_1} \rangle_{\alpha} \times \{\alpha\} \\
&= \limsup_{n\to\infty} \mathrm{end}_{\alpha_{k_1}} u_n^{(k_1)} \cap (X \times \{\alpha\}) \\
&\supseteq \limsup_{n\to\infty} \mathrm{end}_{\alpha_{k_2}} u_n^{(k_2)} \cap (X \times \{\alpha\}) \\
&= \langle v^{k_2} \rangle_{\alpha} \times \{\alpha\}.
\end{aligned} \tag{4}$$

Hence by (3) and (4), $\langle v^{k_1} \rangle_{\alpha} = \langle v^{k_2} \rangle_{\alpha}$ for $\alpha \in (\alpha_{k_1}, 1]$. So (ii) is true. (iii) follows immediately from (i) and (ii).

Define a subset v of $X \times [0, 1]$ given by

$$v = \cup_{k=1}^{+\infty} v^k \cup (X \times \{0\}). \tag{5}$$

From affirmation (b), we can see that

$$\langle v\rangle_\alpha = \begin{cases} \langle v^k\rangle_\alpha, & \text{if for some } k\in\mathbb{N},\ \alpha>\alpha_k, \\ X, & \text{if } \alpha=0, \end{cases} \tag{6}$$

and hence

$$v\cap(X\times(\alpha_k,1]) = v^k\cap(X\times(\alpha_k,1]) \subseteq v^k. \tag{7}$$

We show that $v\in C(X\times[0,1])$. To this end, let $\{(x_l,\gamma_l)\}$ be a sequence in v which converges to an element (x,γ) in $X\times[0,1]$. If $\gamma=0$, then clearly $(x,\gamma)\in v$. If $\gamma>0$, then there is a $k_0\in\mathbb{N}$ such that $\gamma>\alpha_{k_0}$. Hence there is an L such that $\gamma_l>\alpha_{k_0}$ when $l\geq L$. So by (7), $(x_l,\gamma_l)\in v^{k_0}$ when $l\geq L$. Since $v^{k_0}\in K(X\times[\alpha_{k_0},1])$, it follows that $(x,\gamma)\in v^{k_0}\subset v$.

We claim that

(c) $\lim_{n\to\infty} H(\mathrm{end}\, u_n^{(n)}, v)=0$ and $v\in F_{USCG}(X)^e$.

Let $n\in\mathbb{N}$ and $k\in\mathbb{N}$. Then by (5),

$$\begin{aligned} H^*(\mathrm{end}\, u_n^{(n)}, v) &= \max\{H^*(\mathrm{end}_{\alpha_k} u_n^{(n)}, v),\ H^*(\mathrm{end}_0^{\alpha_k} u_n^{(n)}, v)\} \\ &\leq \max\{H^*(\mathrm{end}_{\alpha_k} u_n^{(n)},\ v^k),\ \alpha_k\}, \end{aligned} \tag{8}$$

and by (7),

$$\begin{aligned} H^*(v, \mathrm{end}\, u_n^{(n)}) &= \max\{\sup_{(x,\gamma)\in v\cap(X\times(\alpha_k,1]} \overline{d}((x,\gamma), \mathrm{end}\, u_n^{(n)}),\ H^*(v\cap(X\times[0,\alpha_k]), \mathrm{end}\, u_n^{(n)})\} \\ &\leq \max\{H^*(v^k,\ \mathrm{end}_{\alpha_k} u_n^{(n)}),\ \alpha_k\}. \end{aligned} \tag{9}$$

Clearly, (8) and (9) imply that

$$H(\mathrm{end}\, u_n^{(n)},\ v) \leq \max\{H(\mathrm{end}_{\alpha_k} u_n^{(n)},\ v^k),\ \alpha_k\}. \tag{10}$$

Now we show that

$$\lim_{n\to\infty} H(\mathrm{end}\, u_n^{(n)}, v) = 0. \tag{11}$$

To see this, let $\varepsilon>0$. Notice that $\alpha_k\to 0$ and for each α_k, $k=1,2,\ldots$, $\lim_{n\to\infty} H(\mathrm{end}_{\alpha_k} u_n^{(n)}, v^k)=0$. Then there is an α_{k_0} and an $N\in\mathbb{N}$ such that $\alpha_{k_0}<\varepsilon$ and $H(\mathrm{end}_{\alpha_{k_0}} u_n^{(n)}, v^{k_0})<\varepsilon$ for all $n\geq N$. Thus by (10), $H(\mathrm{end}\, u_n^{(n)}, v)<\varepsilon$ for all $n\geq N$. So (11) is true.

Since the sequence $\{\mathrm{end}\, u_n^{(n)}\}$ is in $F_{USC}(X)^e$ and $\{\mathrm{end}\, u_n^{(n)}\}$ converges to v in $(C(X\times[0,1]),H)$, by Proposition 4.7, it follows that $v\in F_{USC}(X)^e$.

Let $k\in\mathbb{N}$. Then $v^k\in K(X\times[\alpha_k,1])$, and hence $\langle v^k\rangle_\alpha\in K(X)\cup\{\emptyset\}$ for all $\alpha\in[0,1]$. So from (6), $\langle v\rangle_\alpha\in K(X)\cup\{\emptyset\}$ for all $\alpha\in(0,1]$, and thus $v\in F_{USCG}(X)^e$.

From affirmation (c), we have that $\{u_n^{(n)}\}$ is a convergent sequence in $(F_{USCG}(X),H_{\mathrm{end}})$. Note that $\{u_n^{(n)}\}$ is a subsequence of $\{u_n\}$. Thus the proof is completed.

□

Theorem 4.12. *Let U be a subset of $F_{USCG}(X)$. Then U is totally bounded in $(F_{USCG}(X),H_{\mathrm{end}})$ if and only if $U(\alpha)$ is totally bounded in (X,d) for each $\alpha\in(0,1]$.*

Proof. ***Necessity***. The necessity part is Lemma 4.9.

Sufficiency. Suppose that $U(\alpha)$ is totally bounded in (X,d) for each $\alpha\in(0,1]$. Then $U(\alpha)$ is relatively compact in $(\widetilde{X},\widetilde{d})$ for each $\alpha\in(0,1]$. Thus by Theorem 4.11, U is relatively compact in $(F_{USCG}(\widetilde{X}),H_{\mathrm{end}})$. Hence U is totally bounded in $(F_{USCG}(\widetilde{X}),H_{\mathrm{end}})$. So clearly U is totally bounded in $(F_{USCG}(X),H_{\mathrm{end}})$.

□

Theorem 4.13. *Let U be a subset of $F_{USCG}(X)$. Then the following are equivalent:*

(i) U is compact in $(F_{USCG}(X),H_{\mathrm{end}})$;

(ii) $U(\alpha)$ is relatively compact in (X,d) for each $\alpha\in(0,1]$ and U is closed in $(F_{USCG}(X),H_{\mathrm{end}})$;

(iii) $U(\alpha)$ is compact in (X,d) for each $\alpha\in(0,1]$ and U is closed in $(F_{USCG}(X),H_{\mathrm{end}})$.

Proof. By Theorem 4.11, (i) $\Leftrightarrow$ (ii). Obviously (iii) $\Rightarrow$ (ii). We shall complete the proof by showing that (ii) $\Rightarrow$ (iii). Suppose that (ii) is true. To verify (iii), it suffices to show that $U(\alpha)$ is closed in (X,d) for each $\alpha\in(0,1]$. To do this, let $\alpha\in(0,1]$ and let $\{x_n\}$ be a sequence in $U(\alpha)$ with $\{x_n\}$ converges to an element x in (X,d). We only need to show that $x\in U(\alpha)$.

Pick a sequence $\{u_n\}$ in U such that $x_n\in[u_n]_\alpha$ for $n=1,2,\ldots$, which means that $(x_n,\alpha)\in\mathrm{end}\, u_n$ for $n=1,2,\ldots$.

From the equivalence of (i) and (ii), U is compact in $(F_{USCG}(X), H_{\mathrm{end}})$. So there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in U$ such that $H_{\mathrm{end}}(u_{n_k}, u) \to 0$. Hence by Remark 2.4, $\lim^{(\Gamma)}_{n\to\infty} u_{n_k} = u$. Note that $(x, \alpha) = \lim_{k\to\infty}(x_{n_k}, \alpha)$. Thus

$$(x, \alpha) \in \liminf_{n\to\infty} \operatorname{end} u_{n_k} = \operatorname{end} u.$$

So $x \in [u]_\alpha$, and therefore $x \in U(\alpha)$.

It can be seen that Theorem 7.11 in [19] can be verified in a similar manner to that in this theorem.

□

We [18] gave the following characterizations of compactness in $(F_{USCG}(\mathbb{R}^m), H_{\mathrm{end}})$.

Theorem 4.14. *(Theorem 7.1 in [18]) Let U be a subset of $F_{USCG}(\mathbb{R}^m)$. Then U is a relatively compact set in $(F_{USCG}(\mathbb{R}^m), H_{\mathrm{end}})$ if and only if $U(\alpha)$ is a bounded set in $\mathbb{R}^m$ when $\alpha \in (0, 1]$.*

Theorem 4.15. *(Theorem 7.3 in [18]) Let U be a subset of $F_{USCG}(\mathbb{R}^m)$. Then U is a totally bounded set in $(F_{USCG}(\mathbb{R}^m), H_{\mathrm{end}})$ if and only if, for each $\alpha \in (0, 1]$, $U(\alpha)$ is a bounded set in $\mathbb{R}^m$.*

Theorem 4.16. *(Theorem 7.2 in [18]) U is a compact set in $(F_{USCG}(\mathbb{R}^m), H_{\mathrm{end}})$ if and only if U is a closed set in $(F_{USCG}(\mathbb{R}^m), H_{\mathrm{end}})$ and $U(\alpha)$ is a bounded set in $\mathbb{R}^m$ when $\alpha \in (0, 1]$.*

Let S be a set in $\mathbb{R}^m$. Then the following properties are equivalent.
(i) S is a bounded set in $\mathbb{R}^m$.
(ii) S is a totally bounded set in $\mathbb{R}^m$.
(iii) S is a relatively compact set in $\mathbb{R}^m$.

Using the above well-known fact, we can see that Theorem 4.11 implies Theorem 4.14; Theorem 4.12 implies Theorem 4.15; Theorem 4.13 implies Theorem 4.16.

So the characterizations of relative compactness, total boundedness, and compactness in $(F_{USCG}(\mathbb{R}^m), H_{\mathrm{end}})$ given in our previous work [18] are corollaries of the characterizations of relative compactness, total boundedness, and compactness of $(F_{USCG}(X), H_{\mathrm{end}})$ given in this section, respectively.

Furthermore, the characterizations of relative compactness, total boundedness, and compactness of $(F_{USCG}(X), H_{\mathrm{end}})$ given in this section illustrate the relationship between relative compactness, total boundedness, and compactness of a set in $F_{USCG}(X)$ and that of the union of its elements' α-cuts.

From above discussions, we can see that the characterizations of relative compactness, total boundedness, and compactness of $(F_{USCG}(X), H_{\rm end})$ given in this section significantly improve the characterizations of relative compactness, total boundedness, and compactness in $(F_{USCG}(\mathbb{R}^m), H_{\rm end})$ given in our previous work [18].

Remark 4.17. The following clauses (i) and (ii) are pointed out in Remark 5.1 of chinaXiv:202107.00011v2 (we submitted it on 2021-07-22).
(i) $(F^1_{USCG}(X), H_{\rm end})$ can be treated as a subspace of $(C(X \times [0, 1]), H)$ by seeing each $u \in F^1_{USCG}(X)$ as its endograph.
(ii) We can discuss the properties of $(F^1_{USCG}(X), H_{\rm end})$ by treating $(F^1_{USCG}(X), H_{\rm end})$ as a subspace of $(C(X\times[0, 1]), H)$. These properties include characterizations of total boundedness, relative compactness and compactness of $(F^1_{USCG}(X), H_{\rm end})$.

In this paper, we treat $(F_{USCG}(X), H_{\rm end})$ as a subspace of $(C(X\times[0, 1]), H)$ to discuss the properties of $(F_{USCG}(X), H_{\rm end})$.

At the end of this section, we illustrate that Theorems 4.2, 4.1 and 4.3 can be seen as special cases of Theorems 4.11, 4.12, and 4.13, respectively. We begin with some propositions.

The following proposition follows immediately from the basic definitions.

Proposition 4.18. *Let A be a subset of X.*
(i) The conditions (i-1) A is a set in $C(X)$, and (i-2) χ_A is a fuzzy set in $F_{USC}(X)$, are equivalent.
(ii) The conditions (ii-1) A is a set in $K(X)$, (ii-2) χ_A is a fuzzy set in $F_{USCB}(X)$, and (ii-3) χ_A is a fuzzy set in $F_{USCG}(X)$, are equivalent.

Let $\mathcal{D} \subseteq P(X)$. We use the symbol $\mathcal{D}_{F(X)}$ to denote the set $\{C_{F(X)} : C \in \mathcal{D}\}$.

Let $A, B \in C(X)$. Then

$$H_{\rm end}(\chi_A, \chi_B) = \min\{H(A, B),\ 1\}. \tag{12}$$

Proposition 4.19. *Let $\mathcal{D}$ be a subset of $K(X)$.*
(i) $\mathcal{D}$ is totally bounded in $(K(X), H)$ if and only if $\mathcal{D}_{F(X)}$ is totally bounded in $(F_{USCG}(X), H_{\rm end})$;
(ii) $\mathcal{D}$ is compact in $(K(X), H)$ if and only if $\mathcal{D}_{F(X)}$ is compact in $(F_{USCG}(X), H_{\rm end})$.

Proof. From (12), it follows immediately that (i) is true.

By (12), we have that $\mathcal{D}$ is compact in $(K(X), H)$ if and only if $\mathcal{D}_{F(X)}$ is compact in $(K(X)_{F(X)}, H_{\text{end}})$. Clearly $\mathcal{D}_{F(X)}$ is compact in $(K(X)_{F(X)}, H_{\text{end}})$ if and only if $\mathcal{D}_{F(X)}$ is compact in $(F_{USCG}(X), H_{\text{end}})$. So (ii) is true.

□

Proposition 4.20. *Let $\{A_n\}$ be a sequence of sets in $C(X)$. If $\{\chi_{A_n}\}$ converges to a fuzzy set u in $F_{USC}(X)$ according to the H_{end} metric, then there is an $A \in C(X)$ such that $u = \chi_A$ and $H(A_n, A) \to 0$ as $n \to \infty$.*

Proof. We will show in turn, the following properties (i), (ii) and (iii).
(i) Let $x \in X$ and $\alpha, \beta \in (0, 1]$. Then $(x, \alpha) \in \operatorname{end} u$ if and only if $(x, \beta) \in \operatorname{end} u$.
(ii) $[u]_\alpha = [u]_\beta$ for all $\alpha, \beta \in [0, 1]$.
(iii) There is an A in $C(X)$ such that $u = \chi_A$ and $H(A_n, A) \to 0$ as $n \to \infty$.

To show (i), we only need to show that if $(x, \alpha) \in \operatorname{end} u$ then $(x, \beta) \in \operatorname{end} u$ since α and β can be interchanged.

Assume that $(x, \alpha) \in \operatorname{end} u$. Since $H_{\text{end}}(\chi_{A_n}, u) \to 0$, by Theorem 2.3 and Remark 2.4, $\lim^{(\Gamma)}_{n\to\infty} \chi_{A_n} = u$. Then there is a sequence $\{(x_n, \alpha_n)\}$ such that $(x_n, \alpha_n) \in \operatorname{end} \chi_{A_n}$ for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \overline{d}((x_n, \alpha_n), (x, \alpha)) = 0$. As $\alpha > 0$, it follows that there exists an N such that $\alpha_n > 0$ for all $n \geq N$. This yields that $(x_n, \alpha_n) \in \operatorname{send} \chi_{A_n} = A_n \times [0, 1]$ for all $n \geq N$. Hence $(x_n, \beta) \in \operatorname{send} \chi_{A_n}$ for all $n \geq N$. Observe that $\lim_{n\to\infty} \overline{d}((x_n, \beta), (x, \beta)) = 0$, i.e. $\{(x_n, \beta) : n \geq N\}$ converges to (x, β) in $(X \times [0, 1], \overline{d})$. Thus we have $(x, \beta) \in \liminf_{n\to\infty} \operatorname{end} \chi_{A_n} = \operatorname{end} u$. So (i) is true.

From (i), we have that $[u]_\alpha = [u]_\beta$ for all $\alpha, \beta \in (0, 1]$. Then $[u]_0 = \overline{\cup_{\alpha>0}[u]_\alpha} = [u]_1$. So (ii) is true.

Set $A = [u]_1$. By Proposition 5.1, $u \in F'^{1}_{USC}(X)$. From this and (ii), it follows that $A \in C(X)$ and $u = \chi_A$.

Since by (12),

$$H_{\text{end}}(\chi_{A_n}, u) = H_{\text{end}}(\chi_{A_n}, \chi_A) = \min\{H(A_n, A),\ 1\} \to 0 \text{ as } n \to \infty,$$

we obtain that $H(A_n, A) \to 0$ as $n \to \infty$. So (iii) is true. This completes the proof.

□

Proposition 4.21. *Let $\{x_n\}$ be a sequence in X. If $\{\widehat{x_n}\}$ converges to a fuzzy set u in $F_{USC}(X)$ according to the H_{end} metric, then there is an $x \in X$ such that $u = \widehat{x}$ and $d(x_n, x) \to 0$ as $n \to \infty$.*

Proof. Note that $\widehat{z} = \chi_{\{z\}}$ for each $z \in X$. Thus by Proposition 4.20, there is an $A \in C(X)$ such that $u = \chi_A$ and $H(\{x_n\}, A) \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty}^{(K)} \{x_n\} = A$, it follows that A is a singleton. Set $A = \{x\}$. Then $u = \widehat{x}$ and $d(x_n, x) = H(\{x_n\}, \{x\}) \to 0$ as $n \to \infty$. This completes the proof.

□

Proposition 4.21 is Proposition 8.15 in our paper arXiv:submit/4644498. It can be seen that we can also use the idea in the proof of Proposition 4.20 to show Proposition 4.21 directly.

It can be seen that using the idea in the proof of Proposition 8.15 in arXiv:submit/4644498, we can show that A in the proof of Proposition 4.21 is a singleton as follows.

Assume that A has at least two distinct elements. Pick p, q in A with $p \neq q$. Let $z \in X$. Since $d(p, z)+d(q, z) \geq d(p, q)$, it follows that $\max\{d(p, z), d(q, z)\} \geq \frac{1}{2}d(p, q)$. Thus $H(A, \{x_n\}) = H^*(A, \{x_n\}) \geq \frac{1}{2}d(p, q)$, which contradicts $H(A, \{x_n\}) \to 0$ as $n \to \infty$.

Proposition 4.22. *Let $\mathcal{D}$ be a subset of $K(X)$ and $\mathcal{B}$ a subset of $C(X)$.*
(i) $C(X)_{F(X)}$ is closed in $(F_{USC}(X), H_{\text{end}})$.
(ii) $K(X)_{F(X)}$ is closed in $(F_{USCG}(X), H_{\text{end}})$.
(iii) $\mathcal{D}$ is closed in $(K(X), H)$ if and only if $\mathcal{D}_{F(X)}$ is closed in $(F_{USCG}(X), H_{\text{end}})$.
(iv) $\mathcal{D}$ is relatively compact in $(K(X), H)$ if and only if $\mathcal{D}_{F(X)}$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$.
(v) $\mathcal{B}$ is closed in $(C(X), H)$ if and only if $\mathcal{B}_{F(X)}$ is closed in $(F_{USC}(X), H_{\text{end}})$.
(vi) $\mathcal{B}$ is relatively compact in $(C(X), H)$ if and only if $\mathcal{B}_{F(X)}$ is relatively compact in $(F_{USC}(X), H_{\text{end}})$.

Proof. From Proposition 4.20 we have that (i) is true.

By (i), the closure of $K(X)_{F(X)}$ in $(F_{USCG}(X), H_{\text{end}})$ is contained in $F_{USCG}(X) \cap C(X)_{F(X)}$. From Proposition 4.18 (ii), $F_{USCG}(X) \cap C(X)_{F(X)} = K(X)_{F(X)}$. Thus the closure of $K(X)_{F(X)}$ in $(F_{USCG}(X), H_{\text{end}})$ is $K(X)_{F(X)}$. So (ii) is true.

Suppose the following conditions: (a-1) $\mathcal{D}$ is closed in $(K(X), H)$, (a-2) $\mathcal{D}_{F(X)}$ is closed in $(K(X)_{F(X)}, H_{\text{end}})$, and (a-3) $\mathcal{D}_{F(X)}$ is closed in $(F_{USCG}(X), H_{\text{end}})$.

By (12), (a-1)⇔(a-2). From (ii), (a-2)⇔(a-3). Thus (a-1)⇔(a-3). So (iii) is true.

Suppose the following conditions: (b-1) $\mathcal{D}$ is relatively compact in $(K(X), H)$, (b-2) $\mathcal{D}_{F(X)}$ is relatively compact in $(K(X)_{F(X)}, H_{\text{end}})$, and (b-3) $\mathcal{D}_{F(X)}$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$.

By (12), (b-1)$\Leftrightarrow$(b-2). From (ii), (b-2)$\Leftrightarrow$(b-3). Thus (b-1)$\Leftrightarrow$(b-3). So (iv) is true.

Using (12) and (i), (v) and (vi) can be proved in a similar manner to (iii) and (iv), respectively.

$\square$

Each subset $\mathcal{D}$ of $(K(X), H)$ corresponds a subset $\mathcal{D}_{F(X)}$ of $(F_{USCG}(X), H_{\text{end}})$. Using Theorems 4.11, 4.12, and 4.13, we obtain the characterizations of relative compactness, total boundedness, and compactness for $\mathcal{D}_{F(X)}$ in $(F_{USCG}(X), H_{\text{end}})$ as follows.

Corollary 4.23. *Let $\mathcal{D}$ be a subset of $K(X)$. Then $\mathcal{D}_{F(X)}$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$ if and only if $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d).*

Corollary 4.24. *Let $\mathcal{D}$ be a subset of $K(X)$. Then $\mathcal{D}_{F(X)}$ is totally bounded in $(F_{USCG}(X), H_{\text{end}})$ if and only if $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is totally bounded in (X, d).*

Corollary 4.25. *Let $\mathcal{D}$ be a subset of $K(X)$. Then the following are equivalent:*
(i) $\mathcal{D}_{F(X)}$ is compact in $(F_{USCG}(X), H_{\text{end}})$;
(ii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d) and $\mathcal{D}_{F(X)}$ is closed in $(F_{USCG}(X), H_{\text{end}})$;
(iii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is compact in (X, d) and $\mathcal{D}_{F(X)}$ is closed in $(F_{USCG}(X), H_{\text{end}})$.

From Proposition 4.19 and clauses (iii) and (iv) of Proposition 4.22, we obtain that Corollaries 4.23, 4.24 and 4.25 are equivalent forms of Theorems 4.2, 4.1 and 4.3, respectively. So we can see Theorems 4.2, 4.1 and 4.3 as special cases of Theorems 4.11, 4.12, and 4.13, respectively.

5. Characterizations of compactness in $(F^r_{USCG}(X), H_{\text{end}})$

In this section, we first investigate the properties of the H_{end} metric. Then based on the characterizations of relative compactness, total boundedness and compactness in $(F_{USCG}(X), H_{\text{end}})$ given in Section 4, we give characterizations of relatively compact sets, totally bounded sets, and compact sets in $(F^r_{USCG}(X), H_{\text{end}})$, $r \in [0, 1]$. $(F^r_{USCG}(X), H_{\text{end}})$, $r \in [0, 1]$ are a kind of subspaces of $(F_{USCG}(X), H_{\rm end})$. Each element in $F^r_{USCG}(X)$ takes r as its maximum value. $(F^1_{USCG}(X), H_{\rm end})$ is one of these subspaces.

For $D \subseteq X \times [0,1]$, define $\boldsymbol{S_D} := \sup\{\alpha : (x,\alpha) \in D\}$.

We claim that for $D, E \in C(X \times [0,1])$,

$$H(D,E) \geq |S_D - S_E|. \tag{13}$$

To see this, let $D, E \in C(X \times [0,1])$. If $|S_D - S_E| = 0$, then (13) is true. If $|S_D - S_E| > 0$. Assume that $S_D > S_E$. Note that for each $(x,t) \in D$ with $t > S_E$, $\overline{d}((x,t), E) \geq t - S_E$. Thus $H(D,E) \geq \sup\{t - S_E : (x,t) \in D \text{ with } t > S_E\} = S_D - S_E$. So (13) is true.

Let $u \in F(X)$. Define $\boldsymbol{S_u} := \sup\{u(x) : x \in X\}$. We can see that $S_u = S_{\text{end}\, u}$. Clearly $[u]_{S_u} = \emptyset$ is possible.

From (13), we have that for $u, v \in F_{USC}(X)$,

$$H_{\rm end}(u,v) \geq |S_u - S_v|. \tag{14}$$

Proposition 5.1. *Let u and u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F_{USC}(X)$. If $H_{\rm end}(u_n, u) \to 0$ as $n \to \infty$, then $S_{u_n} \to S_u$ as $n \to \infty$.*

Proof. The desired result follows immediately from (14).

□

Let $u \in F(X)$. $\max\{u(x) : x \in X\}$ may not exist. If $\max\{u(x) : x \in X\}$ exists, then obviously $S_u = \max\{u(x) : x \in X\}$. If $[u]_{S_u} \neq \emptyset$, then, as $S_u = \sup\{u(x) : x \in X\}$, it follows that $S_u = \max\{u(x) : x \in X\}$.

Proposition 5.2. *(i) Let $u \in F_{USC}(X)$. If there is an $\alpha \in [0, S_u]$ with $[u]_\alpha \in K(X)$, then $[u]_{S_u} \neq \emptyset$ and $S_u = \max\{u(x) : x \in X\}$.*

(ii) Let $u \in F_{USCG}(X)$. Then $S_u = \max\{u(x) : x \in X\}$.

Proof. First, we show (i). If $\alpha = S_u$, then $[u]_{S_u} \neq \emptyset$. If $\alpha < S_u$, then pick a sequence $\{x_n\}$ in $[u]_\alpha$ with $u(x_n) \to S_u$. From the compactness of $[u]_\alpha$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point x in $[u]_\alpha$. Thus $u(x) \geq \lim_{k\to\infty} u(x_{n_k}) = S_u$. Hence $u(x) = S_u$, and therefore $[u]_{S_u} \neq \emptyset$ and $S_u = \max\{u(x) : x \in X\}$. So (i) is true.

In the $\alpha < S_u$ case, we can also prove $[u]_{S_u} \neq \emptyset$ as follows. Take an increasing sequence $\{\alpha_k\}$ in $[\alpha, 1]$ with $\alpha_k \to S_u-$. Then $[u]_{\alpha_k} \in K(X)$ for each $k = 1, 2, \ldots$, and thus $[u]_{S_u} = \cap_{k=1}^{+\infty} [u]_{\alpha_k} \in K(X)$. So $[u]_{S_u} \neq \emptyset$.

Now we show (ii). If $u \in F_{USCG}(X) \setminus \{\emptyset_{F(X)}\}$, then there exists an $\alpha \in (0, S_u]$ such that $[u]_\alpha \in K(X)$. So from (i), $S_u = \max\{u(x) : x \in X\}$. If

$u = \emptyset_{F(X)}$, then $S_u = 0 = \max\{u(x) : x \in X\}$. So for each $u \in F_{USCG}(X)$, $S_u = \max\{u(x) : x \in X\}$.

□

Let $r \in [0,1]$. Define

$$F'^{r}_{USC}(X) = \{u \in F_{USC}(X) : S_u = r\},$$
$$F^{r}_{USC}(X) = \{u \in F_{USC}(X) : r = \max\{u(x) : x \in X\}\},$$
$$F'^{r}_{USCG}(X) = \{u \in F_{USCG}(X) : S_u = r\},$$
$$F^{r}_{USCG}(X) = \{u \in F_{USCG}(X) : r = \max\{u(x) : x \in X\}\},$$
$$F'^{r}_{USCB}(X) = \{u \in F_{USCB}(X) : S_u = r\},$$
$$F^{r}_{USCB}(X) = \{u \in F_{USCB}(X) : r = \max\{u(x) : x \in X\}\}.$$

Let $r \in [0,1]$. We can see that $F^{r}_{USC}(X) \subseteq F'^{r}_{USC}(X)$. Clearly, $F'^{0}_{USC}(X) = F^{0}_{USC}(X) = F^{0}_{USCG}(X) = F^{0}_{USCB}(X) = \{\emptyset_{F(X)}\}$.

Proposition 5.3. *Let $r \in [0,1]$. Then*
(i) $F^{r}_{USCG}(X) = F'^{r}_{USCG}(X)$, $F^{r}_{USCB}(X) = F'^{r}_{USCB}(X)$,
(ii) $F'^{r}_{USC}(X)$ *is a closed subset of* $(F_{USC}(X), H_{\rm end})$,
(iii) $F^{r}_{USCG}(X)$ *is a closed subset of* $(F_{USCG}(X), H_{\rm end})$, *and*
(iv) $F^{r}_{USCB}(X)$ *is a closed subset of* $(F_{USCB}(X), H_{\rm end})$.

Proof. From Proposition 5.2 (ii) and the fact that $F_{USCB}(X) \subseteq F_{USCG}(X)$, we have that $F^{r}_{USCG}(X) = F'^{r}_{USCG}(X)$, $F^{r}_{USCB}(X) = F'^{r}_{USCB}(X)$. So (i) is true.

By Proposition 5.1, (ii) is true.

From Proposition 5.1, $F'^{r}_{USCG}(X)$ is a closed subset of $(F_{USCG}(X), H_{\rm end})$, and $F'^{r}_{USCB}(X)$ is a closed subset of $(F_{USCB}(X), H_{\rm end})$. Then by (i), (iii) and (iv) are true.

□

Lemma 5.4. *Let $r \in [0,1]$ and let U be a subset of $F^{r}_{USCG}(X)$. Then the following (i-1) is equivalent to (i-2), and (ii-1) is equivalent to (ii-2).*
(i-1) U *is relatively compact in* $(F_{USCG}(X), H_{\rm end})$.
(i-2) U *is relatively compact in* $(F^{r}_{USCG}(X), H_{\rm end})$.
(ii-1) U *is closed in* $(F_{USCG}(X), H_{\rm end})$.
(ii-2) U *is closed in* $(F^{r}_{USCG}(X), H_{\rm end})$.

Proof. Clause (iii) of Proposition 5.3 says that $F^r_{USCG}(X)$ is a closed subset of $(F_{USCG}(X), H_{\rm end})$. From this we obtain that (i-1)⇔(i-2), and (ii-1)⇔(ii-2). □

In this paper, we suppose that $(r, r] = \emptyset$ for $r \in \mathbb{R}$.

Lemma 5.5. *Let $r \in [0, 1]$ and let U be a subset of $F^r_{USCG}(X)$. Then the following (i-1) is equivalent to (i-2), (ii-1) is equivalent to (ii-2), and (iii-1) is equivalent to (iii-2).*
(i-1) $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$.
(i-2) $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, r]$.
(ii-1) $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, 1]$.
(ii-2) $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, r]$.
(iii-1) $U(\alpha)$ is compact in (X, d) for each $\alpha \in (0, 1]$.
(iii-2) $U(\alpha)$ is compact in (X, d) for each $\alpha \in (0, r]$.

Proof. Observe that if $\alpha \in (r, 1]$ then $U(\alpha) = \emptyset$. From this we obtain that (i-1)⇔(i-2), (ii-1)⇔(ii-2), and (iii-1)⇔(iii-2).

□

Corollary 5.6. *Let $r \in [0, 1]$ and let U be a subset of $F^r_{USCG}(X)$. Then the following properties are equivalent.*
(i) U is relatively compact in $(F_{USCG}(X), H_{\rm end})$.
(ii) U is relatively compact in $(F^r_{USCG}(X), H_{\rm end})$.
(iii) $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$.
(iv) $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, r]$.

Proof. By Lemma 5.4, (i)⇔(ii). From this and Theorem 4.11, we obtain that (ii)⇔(iii). By Lemma 5.5, (iii)⇔(iv), and the proof is complete.

□

Corollary 5.7. *Let $r \in [0, 1]$ and let U be a subset of $F^r_{USCG}(X)$. Then the following properties are equivalent.*
(i) U is totally bounded in $(F_{USCG}(X), H_{\rm end})$.
(ii) U is totally bounded in $(F^r_{USCG}(X), H_{\rm end})$.
(iii) $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, 1]$.
(iv) $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, r]$.

Proof. Clearly (i)⇔(ii). From this and Theorem 4.12, we obtain that (ii)⇔(iii). By Lemma 5.5, (iii)⇔(iv), and the proof is complete.

□

Corollary 5.8. *Let $r \in [0,1]$ and let U be a subset of $F^r_{USCG}(X)$. Then the following properties are equivalent.*
(i) U is compact in $(F_{USCG}(X), H_{\rm end})$
(ii) U is compact in $(F^r_{USCG}(X), H_{\rm end})$.
(iii) $U(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$ and U is closed in $(F_{USCG}(X), H_{\rm end})$;
(iv) $U(\alpha)$ is compact in (X,d) for each $\alpha \in (0,1]$ and U is closed in $(F_{USCG}(X), H_{\rm end})$.
(v) $U(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,r]$ and U is closed in $(F^r_{USCG}(X), H_{\rm end})$.
(vi) $U(\alpha)$ is compact in (X,d) for each $\alpha \in (0,r]$ and U is closed in $(F^r_{USCG}(X), H_{\rm end})$.

Proof. Clearly (i)⇔(ii). From this and Theorem 4.13, we obtain that (ii)⇔(iii)⇔(iv).

By Lemma 5.4, U is closed in $(F_{USCG}(X), H_{\rm end})$ if and only if U is closed in $(F^r_{USCG}(X), H_{\rm end})$. By Lemma 5.5, $U(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$ if and only if $U(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,r]$. So (iii)⇔(v).

Similarly, from Lemmas 5.4 and 5.5, we have that (iv)⇔(vi).

So (i)⇔(ii)⇔(iii)⇔(iv)⇔(v)⇔(vi).

□

Remark 5.9. From Corollary 5.8, Lemmas 5.4 and 5.5, the following properties are equivalent.
(i) U is compact in $(F_{USCG}(X), H_{\rm end})$.
(ii) U is compact in $(F^r_{USCG}(X), H_{\rm end})$.
(iii) At least one of (i-1),(i-2), (iii-1) and (iii-2) in Lemma 5.5 holds, and at least one of (ii-1) and (ii-2) in Lemma 5.4 holds.
(iv) All of (i-1),(i-2), (iii-1) and (iii-2) in Lemma 5.5 hold, and all of (ii-1) and (ii-2) in Lemma 5.4 hold.

6. An application on relationship between $H_{\rm end}$ metric and Γ-convergence

As an application of the characterizations of relative compactness, total boundedness and compactness given in Section 4, we discuss the relationship between $H_{\rm end}$ metric and Γ-convergence on fuzzy sets.

Proposition 6.1. *Let S be a nonempty subset of $F_{USC}(X)$. Let u be a fuzzy set in S, and let $\{u_n\}$ be a fuzzy set sequence in S. Then the following properties are equivalent.*
(i) $H_{\rm end}(u_n, u) \to 0$.

(ii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$*, and* $\{u_n, n = 1, 2, \ldots\}$ *is a relatively compact set in* $(F_{USC}(X), H_{\rm end})$*.*
(iii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$*, and* $\{u_n, n = 1, 2, \ldots\}$ *is a relatively compact set in* $(S, H_{\rm end})$*.*
(iv) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$*, and* $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ *is a compact set in* $(S, H_{\rm end})$*.*
(v) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$*, and* $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ *is a compact set in* $(F_{USC}(X), H_{\rm end})$*.*

Proof. To show (i)⇒(v). Assume that (i) is true. By Theorem 2.3 and Remark 2.4, $\lim^{(\Gamma)}_{n\to\infty} u_n = u$. Clearly $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ is a compact set in $(F_{USC}(X), H_{\rm end})$. So (v) is true.

It can be seen that $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ is a compact set in $(F_{USC}(X), H_{\rm end})$ if and only if $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ is a compact set in $(S, H_{\rm end})$. So (v)⇔(iv).

If $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ is a compact set in $(S, H_{\rm end})$, then $\{u_n, n = 1, 2, \ldots\}$ is relatively compact in $(S, H_{\rm end})$ because $\{u_n, n = 1, 2, \ldots\}$ is a subset of $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$. So (iv)⇒(iii).

Clearly if $\{u_n, n = 1, 2, \ldots\}$ is a relatively compact set in $(S, H_{\rm end})$, then $\{u_n, n = 1, 2, \ldots\}$ is a relatively compact set in $(F_{USC}(X), H_{\rm end})$. So (iii)⇒(ii).

To show (ii)⇒(i), we proceed by contradiction. Assume that (ii) is true. If (i) is not true; that is, $H_{\rm end}(u_n, u) \not\to 0$. Then there is an $\varepsilon > 0$ and a subsequence $\{v_n^{(1)}\}$ of $\{u_n\}$ that

$$H_{\rm end}(v_n^{(1)}, u) \geq \varepsilon \text{ for all } n = 1, 2, \ldots. \tag{15}$$

Since $\{u_n, n = 1, 2, \ldots\}$ is relatively compact in $(F_{USC}(X), H_{\rm end})$, there is a subsequence $\{v_n^{(2)}\}$ of $\{v_n^{(1)}\}$ and $v \in F_{USC}(X)$ such that $H_{\rm end}(v_n^{(2)}, v) \to 0$. Hence by Theorem 2.3 and Remark 2.4, $\lim^{(\Gamma)}_{n\to\infty} v_n^{(2)} = v$. Since $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, then by Remark 2.5, $u = v$. So $H_{\rm end}(v_n^{(2)}, u) \to 0$, which contradicts (15).

Since we have shown (i)⇒(v), (v)⇔(iv), (iv)⇒(iii), (iii)⇒(ii) and (ii)⇒(i), the proof is complete.

We can also show this theorem as follows. First we show that (i)⇔(iii) ⇔(iv) by verifying that (i)⇒(iv) ⇒(iii)⇒(i) (The proof of (i)⇒(iv) is similar to that of (i)⇒(v). The proof of (iii)⇒(i) is similar to that of (ii)⇒(i)). Then put $S = F_{USC}(X)$, we obtain that (i)⇔(ii) from (i)⇔(iii), and that (i)⇔(v) from (i)⇔(iv). So we have that (i), (ii), (iii), (iv) and (v) are equivalent to each other.

$\square$

Proposition 6.2. *Let u be a fuzzy set in $F_{USCG}(X)$, and let $\{u_n\}$ be a fuzzy set sequence in $F_{USCG}(X)$. Then the following properties are equivalent.*
(i) $H_{\rm end}(u_n, u) \to 0$.
(ii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, *and for each* $\alpha \in (0, 1]$, $\bigcup_{n=1}^{+\infty}[u_n]_\alpha$ *is relatively compact in* (X, d).
(iii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, *and for each* $\alpha \in (0, 1]$, $\bigcup_{n=1}^{+\infty}[u_n]_\alpha \cup [u]_\alpha$ *is compact in* (X, d).
(iv) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ *is closed in* $(F_{USCG}(X), H_{\rm end})$, *and for each* $\alpha \in (0, 1]$, $\bigcup_{n=1}^{+\infty}[u_n]_\alpha \cup [u]_\alpha$ *is compact in* (X, d).

Proof. The desired result follows from Proposition 6.1, Theorem 4.11 and Theorem 4.13. The proof is routine.

Put $S = F_{USCG}(X)$ in Proposition 6.1. Then we obtain that the following conditions (a), (b) and (c) are equivalent.
(a) $H_{\rm end}(u_n, u) \to 0$.
(b) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, and $\{u_n, n = 1, 2, \ldots\}$ is a relatively compact set in $(F_{USCG}(X), H_{\rm end})$.
(c) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, and $\{u_n, n = 1, 2, \ldots\} \cup \{u\}$ is a compact set in $(F_{USCG}(X), H_{\rm end})$.

(a) is (i). By Theorem 4.11, (b)$\Leftrightarrow$(ii). By Theorem 4.13, (c)$\Leftrightarrow$(iv). We can see that (iv)$\Rightarrow$(iii) $\Rightarrow$(ii). So from (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c), we have that (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv).

$\square$

Proposition 6.3. *Let $\mathcal{D}$ be a nonempty subset of $C(X)$. Let A be a set in $\mathcal{D}$, and let $\{A_n\}$ be a sequence of sets in $\mathcal{D}$. Then the following properties are equivalent.*
(i) $H(A_n, A) \to 0$.
(ii) $\lim^{(K)}_{n\to\infty} A_n = A$, *and* $\{A_n, n = 1, 2, \ldots\}$ *is a relatively compact set in* $(C(X), H)$.
(iii) $\lim^{(K)}_{n\to\infty} A_n = A$, *and* $\{A_n, n = 1, 2, \ldots\}$ *is a relatively compact set in* $(\mathcal{D}, H)$.
(iv) $\lim^{(K)}_{n\to\infty} A_n = A$, *and* $\{A_n, n = 1, 2, \ldots\} \cup \{A\}$ *is a compact set in* $(\mathcal{D}, H)$.
(v) $\lim^{(K)}_{n\to\infty} A_n = A$, *and* $\{A_n, n = 1, 2, \ldots\} \cup \{A\}$ *is a compact set in* $(C(X), H)$.

Proof. The proof is similar to that of Proposition 6.1.

$\square$

Proposition 6.4. *Let A be a set in $K(X)$, and let $\{A_n\}$ be a sequence of sets in $K(X)$. Then the following properties are equivalent.*
(i) $H(A_n, A) \to 0$.
(ii) $\lim^{(K)}_{n\to\infty} A_n = A$, *and* $\bigcup_{n=1}^{+\infty} A_n$ *is a relatively compact set in* (X, d).
(iii) $\lim^{(K)}_{n\to\infty} A_n = A$, *and* $\bigcup_{n=1}^{+\infty} A_n \cup A$ *is a compact set in* (X, d).
(iv) $\lim^{(K)}_{n\to\infty} A_n = A$, $\bigcup_{n=1}^{+\infty} A_n \cup A$ *is a compact set in* (X, d), *and* $\{A_n, n = 1, 2, \ldots\} \cup \{A\}$ *is a closed set in* $(K(X), H)$.

Proof. The desired result follows from Proposition 6.3 and Theorems 4.2 and 4.3. The proof is routine and similar to that of Proposition 6.2.

Put $\mathcal{D} = K(X)$ in Proposition 6.3. Then we obtain that the following conditions (a), (b) and (c) are equivalent.
(a) $H(A_n, A) \to 0$.
(b) $\lim^{(K)}_{n\to\infty} A_n = A$, and $\{A_n, n = 1, 2, \ldots\}$ is a relatively compact set in $(K(X), H)$.
(c) $\lim^{(K)}_{n\to\infty} A_n = A$, and $\{A_n, n = 1, 2, \ldots\} \cup \{A\}$ is a compact set in $(K(X), H)$.

(a) is (i). By Theorem 4.2, (b)$\Leftrightarrow$(ii). By Theorem 4.3, (c)$\Leftrightarrow$(iv). We can see that (iv)$\Rightarrow$(iii) $\Rightarrow$(ii). So from (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c), we have that (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv).

$\square$

Remark 6.5. Let $u \in F_{USCG}(X)$ and $\{u_n\}$ be a fuzzy set sequence in $F_{USC}(X)$. Let $\alpha \in (0, 1]$. Since $[u]_\alpha$ is compact in X, we have that the conditions (a) $\bigcup_{n=1}^{+\infty} [u_n]_\alpha$ is relatively compact in (X, d), and (b) $\bigcup_{n=1}^{+\infty} [u_n]_\alpha \cup [u]_\alpha$ is relatively compact in (X, d), are equivalent.

So "$\bigcup_{n=1}^{+\infty} [u_n]_\alpha$ is relatively compact in (X, d)" can be replaced by "$\bigcup_{n=1}^{+\infty} [u_n]_\alpha \cup [u]_\alpha$ is relatively compact in (X, d)" in clause (ii) of Proposition 6.2.

Similar replacement can be made in Propositions 6.1, 6.3 and 6.4.

Propositions 6.1, 6.2, 6.3 and 6.4 can be shown in different ways. Below we give some other proofs.

Proposition 6.3 implies Proposition 6.1.

Let S be a nonempty subset of $F_{USC}(X)$. Let u be a fuzzy set in S, and let $\{u_n\}$ be a fuzzy set sequence in S.

Put $A = \mathrm{end}\, u$, and for $n = 1, 2, \ldots$, put $A_n = \mathrm{end}\, u_n$ in Proposition 6.3.

Put $\mathcal{D} = \{\mathrm{end}\, u : u \in F_{USC}(X)\}$ in Proposition 6.3. Then from (i)$\Leftrightarrow$(iii) in Proposition 6.3, we obtain that (i)$\Leftrightarrow$(ii) in Proposition 6.1.

Put $\mathcal{D} = \{\text{end}\, u : u \in S\}$ in Proposition 6.3. Then from (i)⇔(iii) in Proposition 6.3, we obtain that (i)⇔(iii) in Proposition 6.1.

Similarly, we can show that (i)⇔(iv) and (i)⇔(v) in Proposition 6.1. So Proposition 6.3 implies Proposition 6.1.

Proposition 6.3, Theorem 4.11 and Theorem 4.13 imply Proposition 6.2.

Let u be a fuzzy set in $F_{USCG}(X)$, and let $\{u_n\}$ be a fuzzy set sequence in $F_{USCG}(X)$.

Put $A = \text{end}\, u$, and for $n = 1, 2, \ldots$, put $A_n = \text{end}\, u_n$ in Proposition 6.3.

Put $\mathcal{D} = \{\text{end}\, u : u \in F_{USCG}(X)\}$ in Proposition 6.3. Then from (i)⇔(iii) in Proposition 6.3 and Theorem 4.11, we obtain that (i)⇔(ii) in Proposition 6.2.

Similarly from (i)⇔(iv) in Proposition 6.3 and Theorem 4.13, we obtain that (i)⇔(iv) in Proposition 6.2. Since (iv)⇒(iii) ⇒(ii) in Proposition 6.2, it follows that (i)⇔(ii)⇔(iii)⇔(iv) in Proposition 6.2. So Proposition 6.3, Theorem 4.11 and Theorem 4.13 imply Proposition 6.2.

Proposition 6.1 implies Propositions 6.3 and 6.4.

Proposition 6.2 implies Proposition 6.4.

Proposition 6.6. *(i) Let $A \in C(X)$ and $\{A_n\}$ a sequence in $C(X)$. Then $H_{\text{end}}(\chi_{A_n}, \chi_A) \to 0$ if and only if $H(A_n, A) \to 0$.*
(ii) Let $A \in P(X)$ and $\{A_n\}$ a sequence in $P(X)$. Then $\lim^{(\Gamma)}_{n\to\infty} \chi_{A_n} = \chi_A$ if and only if $\lim^{(K)}_{n\to\infty} A_n = A$.
(iii) Let $\mathcal{D}$ be a subset of $C(X)$ and $\mathcal{B}$ a subset of $\mathcal{D}$. Then $\mathcal{B}$ is totally bounded (respectively, relatively compact, compact, closed) in $(\mathcal{D}, H)$ if and only if $\mathcal{B}_{F(X)}$ is totally bounded(respectively, relatively compact, compact, closed) in $(\mathcal{D}_{F(X)}, H_{\text{end}})$.

Proof. (i) and (iii) follow immediately from (12). (ii) follows from the definition of Kuratowski convergence and Γ-convergence.

□

Let $\mathcal{D}$ be a nonempty subset of $C(X)$. Let A be a set in $C(X)$, and let $\{A_n\}$ be a sequence of sets in $\mathcal{D}$.

Let $S = C(X)_{F(X)}$ in Proposition 6.1. Then from (i)⇔(iii) in Proposition 6.1, we have that:
(c-1) $H_{\text{end}}(\chi_{A_n}, \chi_A) \to 0$ if and only if $\lim^{(\Gamma)}_{n\to\infty} \chi_{A_n} = \chi_A$, and $\{\chi_{A_n}, n = 1, 2, \ldots\}$ is a relatively compact set in $(C(X)_{F(X)}, H_{\text{end}})$.

By Proposition 6.6, (c-1) means that (i)⇔(ii) in Proposition 6.3.

Let $S = \mathcal{D}_{F(X)}$ in Proposition 6.1. Then from (i)⇔(iii) in Proposition 6.1, we have that:
(c-2) $H_{\rm end}(\chi_{A_n}, \chi_A) \to 0$ if and only if $\lim^{(\Gamma)}_{n\to\infty} \chi_{A_n} = \chi_A$, and $\{\chi_{A_n}, n = 1, 2, \ldots\}$ is a relatively compact set in $(\mathcal{D}_{F(X)}, H_{\rm end})$.

By Proposition 6.6, (c-2) means that (i)⇔(iii) in Proposition 6.3.

Similarly, by using Proposition 6.6, we can show that (i)⇔(iv) in Proposition 6.1 implies that (i)⇔(iv) and (i)⇔(v) in Proposition 6.3.

By clauses (i) and (ii) of Proposition 6.6 and clause (iii) of Proposition 4.22, we can see that Proposition 6.2 implies Proposition 6.4.

By using level characterizations of $H_{\rm end}$ and Γ-convergence on fuzzy sets, it is easy to show that Proposition 6.4 also implies Proposition 6.2.

7. Conclusion

In this paper, we present the characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}(X), H_{\rm end})$. Here X is a general metric space. Based on this, we also give the characterizations of total boundedness, relative compactness and compactness in $(F^r_{USCG}(X), H_{\rm end})$, $r \in [0, 1]$. $(F^r_{USCG}(X), H_{\rm end})$, $r \in [0, 1]$, are metric subspaces of $(F_{USCG}(X), H_{\rm end})$.

The conclusions in this paper significantly improve the corresponding conclusions given in our previous paper [18]. Therein we give the characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}(\mathbb{R}^m), H_{\rm end})$. $\mathbb{R}^m$ is a special type of metric space.

We discuss the relationship between $H_{\rm end}$ metric and Γ-convergence as an application of the characterizations of relative compactness, total boundedness and compactness given in this paper.

The results in this paper have potential applications in the research of fuzzy sets involved the endograph metric and the Γ-convergence.

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