A Note on the Invariant Distribution of a Stochastic Dynamical System

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27th April 2023

Abstract

This paper demonstrates the invariant distribution of a stochastic dynamical system. We give the invariant distribution and numerical examples. We also present a further discussion on the computation details.

Keywords Invariant distribution · stochastic dynamical system

1 The stochastic dynamic system and invariant distribution

The stochastic dynamic system we focus on is

\[
\begin{align*}
    dx &= vdt \\
    dv &= -\nabla V dt - \gamma(v - Ax) dt + Av dt + \sigma d\omega,
\end{align*}
\]

where

\[ V = \frac{1}{2} x^T B x. \]

This equation can be written as

\[
d\begin{pmatrix} x \\ v \end{pmatrix} = M \begin{pmatrix} x \\ v \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \sigma d\omega,
\]

where

\[
M = \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix}.
\]
We can obtain the solution of the initial form of the problem is
\[ x(t) = e^{Mt}x(0) + \int_0^t e^{M(t-\tau)} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} d\omega_\tau. \]

**Remark.** We assume that
\[ \lim_{t \to +\infty} e^{Mt} = 0. \]

Then \( x(t) \) will converge to a limit distribution when \( t \to \infty \). It can be proved that it is a normal distribution \( N(0, C) \), where
\[ C = \sigma^2 \int_0^{+\infty} \exp(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp(M^T t) \, dt. \]

Therefore, we obtain a linear system below
\[ MC + CM^T = \sigma^2 \int_0^{+\infty} \left( M \exp(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp(M^T t) + \exp(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp(M^T t) M^T \right) \, dt \]
\[ = \sigma^2 \int_0^{+\infty} \frac{d}{dt} \left( \exp(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp(M^T t) \right) \, dt \]
\[ = -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \]

Then we can obtain the theorem below

**Theorem 1.1.** Suppose that we have an equation
\[
\begin{aligned}
\begin{cases}
\frac{dx}{dt} &= vdt \\
\frac{dv}{dt} &= -\nabla V dt - \gamma(v - Ax) dt + Av dt + \sigma d\omega.
\end{cases}
\end{aligned}
\]

(1.2)

Let
\[ M = \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix}. \]

Suppose that
\[ \lim_{t \to +\infty} e^{Mt} = 0, \]
then the limit distribution is \( N(0, C) \) where
\[ MC + CM^T = -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \]

From this equation we can not only solve the matrix \( C \) easily but also obtain some corollaries. Here we give the most obvious one.
Corollary 1.1. Suppose that
\[ C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}. \]

We have
(1) \( C_2 \) is a skew-symmetric matrix.
(2) the diagonal elements of \( C_2 \) are all 0.

Proof. From
\[ MC + CM^T = -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \]
We can obtain
\[ C_2 + C_3 = 0. \]
But \( C \) is the covariance matrix of a normal distribution, so \( C \) is symmetric. Therefore,
\[ C_3 = C_2^T \]
We conclude
\[ C_2^T = -C_2 \]
So (1) is proved. (2) is a corollary of (1). \( \square \)

2 Numerical examples

We give two examples.

Example 2.1.
\[ n = 2, \sigma = 1, \gamma = 1, A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, B = I. \]
\[ M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & b\gamma & -\gamma & b \\ 0 & -1 & 0 & -\gamma \end{pmatrix}, \]
\[ C = \begin{pmatrix} c_{11} & c_{12} & 0 & c_{14} \\ c_{12} & c_{22} & -c_{14} & 0 \\ 0 & -c_{14} & c_{33} & c_{34} \\ c_{14} & 0 & c_{34} & c_{44} \end{pmatrix}, \]
\[ D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]
\[ MC + CM^T = D. \]
We obtain
\[ c_{11} = \frac{1}{4} (3b^2 + 2), \quad c_{12} = \frac{b}{2}, \quad c_{22} = \frac{1}{2}, \quad c_{33} = \frac{1}{2} (b^2 + 1), \quad c_{34} = \frac{b}{4}, \quad c_{44} = \frac{1}{2}, \quad c_{14} = -\frac{b}{4}. \]

Therefore
\[
 C = \begin{pmatrix}
 \frac{1}{4} (3b^2 + 2) & \frac{b}{2} & 0 & -\frac{b}{4} \\
 \frac{b}{2} & \frac{1}{2} & \frac{b}{4} & 0 \\
 0 & \frac{b}{4} & \frac{1}{2} (b^2 + 1) & \frac{b}{4} \\
 -\frac{b}{4} & 0 & \frac{b}{4} & \frac{1}{2}
\end{pmatrix}.
\]

\[
 \rho = \frac{1}{\sqrt{2\pi c}} \exp \left( -\frac{1}{2} (q_1, q_2, p_1, p_2)^T C^{-1} (q_1, q_2, p_1, p_2) \right).
\]

From the Fokker-Planck equation [1]
\[
 0 = \frac{d\rho}{dt} = \sum_{i=1}^{n} -\frac{\partial}{\partial q_i} (p_i \rho) - \frac{\partial}{\partial p_i} \left[ \left( -\frac{\partial V}{\partial q_i} - \gamma \left( p_i - \sum_j A_{ij} q_j \right) + \sum_j A_{ij} p_j \right) \rho \right] + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial p_i^2}.
\]

we can confirm this result.

**Example 2.2.**

\[ n = 1, A = a, B = b \]

\[ M = \begin{pmatrix} 0 & 1 \\ \gamma a - b & a - \gamma \end{pmatrix} \]

From
\[
 0 = \frac{d\rho}{dt} = \sum_{i=1}^{n} -\frac{\partial}{\partial q_i} (p_i \rho) - \frac{\partial}{\partial p_i} \left[ \left( -\frac{\partial V}{\partial q_i} - \gamma \left( p_i - \sum_j A_{ij} q_j \right) + \sum_j A_{ij} p_j \right) \rho \right] + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial p_i^2},
\]

we can obtain
\[
 C = \begin{pmatrix}
 \frac{\sigma^2}{2 (a - \gamma)^2 (\gamma a - b)} & 0 \\
 0 & -\frac{\sigma^2}{2 (a - \gamma)}
\end{pmatrix},
\]

\[
 \rho = \frac{1}{\sqrt{2\pi c}} \exp \left( -\frac{1}{2} (q, p)^T C^{-1} (q, p)^T \right).
\]

From the Fokker-Planck equation
\[
 0 = \frac{d\rho}{dt} = \sum_{i=1}^{n} -\frac{\partial}{\partial q_i} (p_i \rho) - \frac{\partial}{\partial p_i} \left[ \left( -\frac{\partial V}{\partial q_i} - \gamma \left( p_i - \sum_j A_{ij} q_j \right) + \sum_j A_{ij} p_j \right) \rho \right] + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial p_i^2},
\]

3 The computation analysis

In this section, we take more discussion on
\[
 MC + CM^T = -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
Assume that $C' = \frac{1}{\sigma^2}C$, and in this case, we obtain:

$$MC' + C'M^T = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$ 

For conveniently writing, we denote $C'$ as $C$ in the following

$$MC + CM^T = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where $M$ is known, $C$ is unknown based on simple calculation, we have the conclusion that the total account of the unknown variable is $2n \times 2n$. The system is equivalent to a linear equation set $DX = y$, where $d \in \mathbb{R}^{4n^2 \times 4n^2}$, $x \in \mathbb{R}^{4n^2}$, $y \in \mathbb{R}^{4n^2}$. Since

$$C = \begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix},$$

where $C_1$ is a symmetric matrix, $C_2$ is a skew symmetric matrix, $C_4$ is a symmetric matrix, the amount of the unknown components, in fact, is only

$$\frac{1}{2}n(n+1) \times 2 + \frac{1}{2}n(n-1) = \frac{3}{2}n^2 + \frac{1}{2}n.$$

Then we obtain that

$$MC + CM^T = \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix},$$

$$= \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix} + \begin{pmatrix} C_2 & C_4 \\ -C_2 & C_4 \end{pmatrix} \begin{pmatrix} 0 & P^T \\ I & Q^T \end{pmatrix},$$

$$= \begin{pmatrix} 0 & C_4 + C_1^P + C_2^Q \\ C_4 + C_1^P + C_2^Q \end{pmatrix},$$

where $P = \gamma A - B, Q = A - \gamma I$.

Thus we obtain that

$$\begin{cases} C_4 + C_1^P + C_2^Q = 0, \\ PC_2 + QC_4 - C_2^P + C_4^Q = I. \end{cases}$$

According to the first line of (3.1), it holds that

$$C_4 = -C_1^P - C_2^Q.$$
Besides, we know $C_4$ is symmetric matrix, which means $C_4^T = C_4$. Thus we obtain:

$$
C_4 = -C_1 P^T - C_2 Q^T \\
= -\left( C_1 P^T - C_2 Q^T \right)^T \\
= -PC_1 - QC_2 \\
= -PC_1 - QC_2.
$$

Substitute (3.2) into the left hand side of the second line of (3.1), and then we obtain that

$$
PC_2 + Q \left( -C_1 P^T \right) - C_2 P^T + \left( PC_1 + QC_2 \right) Q^T = -I,
$$

and

$$
PC_2 - PC_1 Q^T - C_2 P^T - QC_1 P^T = -I.
$$

Thus calculating the covariance matrix $C$ which is equivalent to solving (3.1) is finally equivalent to solving the equations

$$
\begin{cases}
C_4 + C_1 P^T + C_2 Q^T = 0 \\
\left( C_2 - C_1 Q^T \right)^T P^T = -I
\end{cases}
$$

(3.3)

without the integral.

**Example 3.1.**

$n = 2, \sigma = 1, \gamma = 1, A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, B = I$.

so

$$
P = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}.
$$

We denote $C_1$ by $C_{ij}^1$, $C_2$ by $C_{ij}^2$ and $C_4$ by $C_{ij}^4$. According to symmetry or skew-symmetry of $C_1, C_2, C_4$, we obtain the linear equations:

$$
-2C_{11}^1 + 2bC_{12}^1 - 2C_{21}^1 + 2b \left( C_{21}^2 - bC_{22}^1 + C_{21}^2 \right) = -1,
$$

$$
-C_{12}^1 - C_{21}^1 + bC_{22}^1 - C_{12}^2 - C_{21}^2 + b \left( C_{22}^2 + C_{22}^2 \right) = 0,
$$

$$
-C_{11}^1 + bC_{12}^1 - C_{11}^2 + bC_{12}^2 + C_{11}^1 = 0,
$$

$$
-C_{21}^1 + bC_{22}^1 - C_{21}^2 + bC_{22}^2 + C_{21}^1 = 0,
$$

$$
-C_{12}^1 - C_{12}^2 + C_{12}^1 = 0,
$$

$$
-C_{22}^1 - C_{22}^2 + C_{22}^1 = 0,
$$

$$
-2C_{22}^1 - 2C_{22}^2 = -1,
$$

$$
C_{12}^4 - C_{21}^4 = 0,
$$

$$
C_{12}^1 - C_{21}^1 = 0.
$$
\[ C_{12}^2 + C_{21}^2 = 0, \]
\[ C_{11}^2 = 0, \]
\[ C_{22}^2 = 0. \]

Solving the equations, we obtain \( C \) as
\[
C = \begin{pmatrix}
\frac{1}{4} (2 + 3b^2) & \frac{b}{2} & 0 & -\frac{b}{4} \\
\frac{b}{2} & \frac{1}{4} & \frac{b}{4} & 0 \\
0 & \frac{b}{4} & \frac{1}{2} (1 + b^2) & \frac{b}{4} \\
-\frac{b}{4} & 0 & \frac{b}{4} & \frac{1}{2}
\end{pmatrix}.
\]

Fokker-Planck Equation helps us to check the solution
\[
\frac{d\rho}{dt} = \sum_{i=1}^{n} -\frac{\partial}{\partial q_i} (p_i \rho) - \frac{\partial}{\partial p_i} \left[ \left( -\frac{\partial V}{\partial q_i} - \gamma \left( p_i - \sum_j A_{ij} q_j \right) \right) + \sum_j A_{ij} p_j \right] \rho + \frac{\sigma^2}{2} \frac{\partial^2}{\partial p_i^2} \rho = 0.
\]

4 Conclusions

This paper demonstrates the invariant distribution of a stochastic dynamical system. We give the invariant distribution and numerical examples. We also give the details of the computation analysis.

Acknowledgement. The author is very grateful to Professor Ya-xiang Yuan for his support, encouragement, and guidance. The author thanks Professor Molei Tao very much for providing the question and the help.

References