Upper bounds for $Z_1$-eigenvalues of generalized Hilbert tensors *

Juan Meng†, Yisheng Song‡

Abstract

In this paper, we introduce the concept of $Z_1$-eigenvalue to infinite dimensional generalized Hilbert tensors (hypermatrix) $H_\lambda^\infty = (H_{i_1 i_2 \cdots i_m})$,

$$H_{i_1 i_2 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots + i_m + \lambda}, \lambda \in \mathbb{R} \setminus \mathbb{Z}^-; i_1, i_2, \cdots, i_m = 0, 1, 2, \cdots, n, \cdots,$$

and proved that its $Z_1$-spectral radius is not larger than $\pi$ for $\lambda > \frac{1}{2}$, and is at most $\frac{\pi}{\sin \frac{\pi}{n}}$ for $\frac{1}{2} \geq \lambda > 0$. Besides, the upper bound of $Z_1$-spectral radius of an $m$th-order $n$-dimensional generalized Hilbert tensor $H_\lambda^n$ is obtained also, and such a bound only depends on $n$ and $\lambda$.

Key words: Infinite-dimensional generalized Hilbert tensor, $Z_1$-eigenvalue, Spectral radius, Hilbert inequalities.

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1 Introduction

A generalized Hilbert matrix has the form [13]:

$$H_\lambda^\infty = \left( \frac{1}{i + j + \lambda} \right)_{i, j \in \mathbb{Z}^+}$$

(1.1)

where $\mathbb{Z}^+$ ($\mathbb{Z}^-$) is the set of all non-negative (non-positive) integers and $\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$. Denote such a Hilbert matrix with $i, j \in I_n = \{0, 1, 2, \cdots, n\}$ by $H_\lambda^n$. When $\lambda = 1$, such a matrix is called Hilbert matrix, which was introduced by Hilbert [12]. Choi [6] and Ingham [14] proved that Hilbert matrix $H_1^\infty$ is a bounded linear operator (but not compact operator).

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†School of Mathematics and Information Science, Henan Normal University, XinXiang HeNan, P.R. China, 453007. Email: 1015791785@qq.com

‡Corresponding author. School of Mathematics and Information Science and Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Henan Normal University, XinXiang HeNan, P.R. China, 453007. Email: songyisheng@htu.cn.
from Hilbert space $l^2$ into itself. Magnus [18] and Kato [15] studied the spectral properties of $H_1^\infty$. Frazer [7] and Taussky [29] discussed some nice properties of $n$-dimensional Hilbert matrix $H^n_1$. Rosenblum [23] showed that for a real $\lambda < 1$, $H_\lambda^\infty$ defines a bounded operator on $l^p$ for $2 < p < \infty$ and that $\pi \sec \pi x$ is an eigenvalue of $H_\lambda^\infty$ for $|\Re u| < \frac{1}{2} - \frac{1}{p}$. For each non-integer complex number $\lambda$, Aleman, Montes-Rodriguez, Sarafianou [1] showed that $H_\lambda^\infty$ defines a bounded linear operator on the Hardy spaces $H^p (1 < p < \infty)$.

As a natural extension of a generalized Hilbert matrix, the generalized Hilbert tensor (hypermatrix) was introduced by Mei and Song [24]. For each $\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$, the entries of an $m$th-order infinite dimensional generalized Hilbert tensor $H_\lambda^\infty = (H_{i_1i_2\cdots i_m})$ are defined by

$$H_{i_1i_2\cdots i_m} = \frac{1}{i_1 + i_2 + \cdots + i_m + \lambda}, \quad i_1, i_2, \cdots, i_m = 0, 1, 2, \cdots, n, \cdots. \quad (1.2)$$

They showed $H_\lambda^\infty$ defines a bounded and positively $(m - 1)$-homogeneous operator from $l^1$ into $l^p (1 < p < \infty)$. Song and Qi [25] studied the operator properties of Hilbert tensors $H_1^\infty$ and the spectral properties of $H_\lambda^\infty$. Such a tensor, $H_\lambda^\infty$ may be refered to as a Hankel tensor with $v = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots)$. The concept of Hankel tensor was introduced by Qi [22]. For more further research of Hankel tensors, see Qi [22], Chen and Qi [5], Xu [31]. Denote such an $m$th-order $n$-dimensional generalized Hilbert tensor by $H_\lambda^\infty$.

For a real vector $x = (x_1, x_2, \cdots, x_n, x_{n+1}, \cdots) \in l^1$, $H_\lambda^\infty x^{m-1}$ is an infinite dimensional vector with its $i$th component defined by

$$(H_\lambda^\infty x^{m-1})_i = \sum_{i_1, i_2, \cdots, i_m = 0}^{\infty} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m + \lambda}, \quad \lambda \in \mathbb{R} \setminus \mathbb{Z}^-; \quad i = 0, 1, 2, \cdots. \quad (1.3)$$

Accordingly, $H_\lambda^\infty x^m$ is given by

$$H_\lambda^\infty x^m = \sum_{i_1, i_2, \cdots, i_m = 0}^{\infty} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m + \lambda}, \quad \lambda \in \mathbb{R} \setminus \mathbb{Z}^- \quad (1.4)$$

Mei and Song [24] proved that $H_\lambda^\infty x^m < \infty$ and $H_\lambda^\infty x^{m-1} \in l^p (1 < p < \infty)$ for all real vector $x \in l^1$.

In this paper, we will introduce the concept of $Z_1$-eigenvalue $\mu$ for an $m$th-order infinite dimensional generalized Hilbert tensor $H_\lambda^\infty$ and will study some upper bounds of $Z_1$-spectral radius for infinite dimensional generalized Hilbert tensor $H_\lambda^\infty$ and $n$-dimensional generalized Hilbert tensor $H_\lambda^n$.

In Section 2, we will give some Lemmas and basic conclusions, and introduce the concept of $Z_1$-eigenvalue. In Section 3, with the help of the Hilbert type inequalities, the upper bound of $Z_1$-spectral radius of $H_\lambda^\infty$ with $\lambda > 0$ is at most $\pi$ when $\lambda > 1$, and is not larger than $\frac{\pi}{\sqrt{\lambda n}}$ when $0 < \lambda \leq \frac{1}{2}$. Furthermore, for each $Z_1$-eigenvalue $\mu$ of $H_\lambda^\infty$, $|\mu|$ is smaller than or equal to $C(n, \lambda)$, where $C(n, \lambda)$ only depends on the structured coefficient $\lambda$ of generalized Hilbert tensor and the dimensionality $n$ of European space.
2 Preliminaries and Basic Results

For $0 < p < \infty$, $l^p$ is a space consisting of all real number sequences $x = (x_i)_{i=1}^{+\infty}$ satisfying $\sum_{i=1}^{+\infty} |x_i|^p < \infty$. If $p \geq 1$, then a norm on $l^p$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^{+\infty} |x_i|^p\right)^{\frac{1}{p}}.$$

It is well known that $l^2$ is a Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{i=0}^{+\infty} x_i y_i.$$

Clearly, $\|x\|_2 = \sqrt{\langle x, x \rangle}$.

For $p \geq 1$, a norm $\mathbb{R}^n$ can be defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

It is well known that

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2. \quad (2.1)$$

The following Hilbert type inequalities were proved by Frazer [7] on $\mathbb{R}^n$ and Ingham [14] on $l^2$, respectively.

**Lemma 2.1.** (Frazer [7]) Let $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. Then

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \frac{|x_i| |x_j|}{i+j+1} \leq \left(n \sin \frac{\pi}{n}\right) \sum_{k=0}^{n} x_k^2 = \|x\|_2^2 \sin \frac{\pi}{n}, \quad (2.2)$$

**Lemma 2.2.** (Ingham [14]) Let $x = (x_1, x_2, \cdots, x_n, \cdots)^T \in l^2$ and $a > 0$. Then

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{|x_i| |x_j|}{i+j+a} \leq M(a) \sum_{k=0}^{\infty} x_k^2 = M(a) \|x\|_2^2, \quad (2.3)$$

where

$$M(a) = \begin{cases} \frac{\pi}{8n \pi^2}, & 0 < a \leq \frac{1}{2}, \\ \pi, & a > \frac{1}{2}. \end{cases}$$

An $m$-order $n$-dimensional tensor (hypermatrix) $\mathbf{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m} \in \mathbb{R}$, where $i_j \in I_n = \{1, 2, \cdots, n\}$ for $j \in [m] = \{1, 2, \cdots, m\}$. We use $T_{m,n}$ to denote the set of all real $m$th-order $n$-dimensional tensors. Then $\mathbf{A} \in T_{m,n}$ is called a symmetric tensor if the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices. $\mathbf{A} \in T_{m,n}$ is called nonnegative (positive) if $a_{i_1 i_2 \cdots i_m} \geq 0 (a_{i_1 i_2 \cdots i_m} > 0)$ for all $i_1, i_2, \cdots, i_m$.  


Definition 2.1. (Chang and Zhang [2]) Let $A \in T_{m,n}$. A number $\mu \in \mathbb{R}$ is called $Z_1$-eigenvalue of $A$ if there is a real vector $x$ such that

$$\begin{align*}
Ax^{m-1} &= \mu x \\
\|x\|_1 &= 1
\end{align*}$$

(2.4)

and call such a vector $x$ an $Z_1$-eigenvector associated with $\mu$.

For the concepts of eigenvalues of higher order tensors, Qi [19, 20] first used and introduced them for symmetric tensors, and Lim [17] independently introduced this notion but restricted $x$ to be a real vector and $\lambda$ to be a real number. Subsequently, the spectral properties of nonnegative matrices had been generalized to $n$-dimensional nonnegative tensors under various conditions by Chang et al. [3, 4], He and Huang [9], He [10], He et al. [11], Li et al. [16], Qi [21], Song and Qi [26, 27], Wang et al. [30], Yang and Yang [32, 33] and references therein. The notion of $Z_1$-eigenvalue was introduced by Chang and Zhang [2] for higher Markov chains. Now we introduce it to infinite dimensional generalized Hilbert tensors.

Let $H_1$ be an $m$th-order infinite dimensional generalized Hilbert tensor. A real number $\mu$ is called a $Z_1$-eigenvalue of $H_1$ if there exists a nonzero vector $x \in \ell^p$ satisfying

$$T_\infty x = \|x\|_1^{2-m}H_1^{\infty}x^{m-1} = \mu x.$$  

(2.5)

where $\theta = (0,0,\cdots,0,\cdots)$. Mei and Song [24] first used the concept of the operator $T_\infty$ induced by a generalized Hilbert tensor $H_1^{\infty}$ and showed $T_\infty$ is a bounded and positively homogeneous operator from $\ell^1$ into $\ell^p$ ($1 < p < \infty$). Then $T_\infty$ is referred to as a bounded and positively homogeneous operator from $\ell^2$ into $\ell^2$. So, the concept of $Z_1$-eigenvalue may be introduced to the infinite dimensional Hilbert tensor $H_1^{\infty}$.

Definition 2.2. Let $H_1^{\infty}$ be an $m$th-order infinite dimensional generalized Hilbert tensor. A real number $\mu$ is called a $Z_1$-eigenvalue of $H_1^{\infty}$ if there exists a nonzero vector $x \in \ell^2$ satisfying

$$T_\infty x = \|x\|_1^{2-m}H_1^{\infty}x^{m-1} = \mu x.$$  

(2.6)

Such a vector $x$ is called an $Z_1$-eigenvector associated with $\mu$.

3 Main Results

Theorem 3.1. Let $H_1^{n}$ be an $m$th-order $n$-dimensional generalized Hilbert tensor. Then

$$|\mu| \leq C(n, \lambda) \text{ for all } Z_1\text{-eigenvalue } \mu \text{ of } H_1^{n},$$

where $[\lambda]$ is the largest integer not exceeding $\lambda$ and

$$C(n, \lambda) = \begin{cases} 
    n\sin \frac{\pi}{n}, & \lambda \geq 1; \\
    \frac{n}{\lambda}, & 1 > \lambda > 0; \\
    \min\{1, \lambda+1\} - \lambda, & -mn < \lambda < 0; \\
    \frac{n}{-mn-\lambda}, & \lambda < -mn.
\end{cases}$$
Proof. For $\lambda \geq 1$, it follows from Lemma 2.1 that for all nonzero vector $x \in \mathbb{R}^n$,
\[
\|H^\lambda x^n\| = \left| \sum_{i_1, i_2, \ldots, i_m = 0}^{n} \frac{x_{i_1}x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m + \lambda} \right| 
\leq \sum_{i_1, i_2, \ldots, i_m = 0}^{n} \frac{|x_{i_1}||x_{i_2}| \cdots |x_{i_m}|}{i_1 + i_2 + \lambda}
= \left( \sum_{i_1 = 0}^{n} \sum_{i_2 = 0}^{n} \frac{|x_{i_1}|}{i_1 + i_2 + \lambda} \right) \sum_{i_3, i_4, \ldots, i_m = 0}^{n} |x_{i_3}| |x_{i_4}| \cdots |x_{i_m}|
\leq \left( \sum_{i_1 = 0}^{n} \sum_{i_2 = 0}^{n} \frac{|x_{i_1}|}{i_1 + i_2 + 1} \right) \sum_{i_3, i_4, \ldots, i_m = 0}^{n} |x_{i_3}| |x_{i_4}| \cdots |x_{i_m}|
\leq \left( \|x\|_2^2 n \sin \frac{\pi}{n} \right)^{m-2} \left( \sum_{i_1 = 0}^{n} |x_i| \right)^{m-2} 
= \|x\|_2^2 \|x\|_1^{m-2} n \sin \frac{\pi}{n}.
\]
That is,
\[
\|H^\lambda x^n\| \leq \|x\|_2^2 \|x\|_1^{m-2} n \sin \frac{\pi}{n}. \tag{3.1}
\]
Since $\mu$ is a $Z_1$-eigenvalue of $H^\lambda$, then there exists a nonzero vector $x$ such that
\[
H^\lambda x^{m-1} = \mu x \text{ and } \|x\|_1 = 1. \tag{3.2}
\]
Thus, we have,
\[
|\mu x^\top x| = |x^\top (H^\lambda x^{m-1})| = \|H^\lambda x^n\| \leq \|x\|_2^2 \|x\|_1^{m-2} n \sin \frac{\pi}{n},
\]
and then,
\[
|\mu| \|x\|_2^2 \leq \|x\|_2^2 \|x\|_1^{m-2} n \sin \frac{\pi}{n}.
\]
As a result,
\[
|\mu| \leq n \sin \frac{\pi}{n}. \tag{3.3}
\]
For all $\lambda \in \mathbb{R} \setminus \mathbb{Z}^-$ with $\lambda < 1$, it is obvious that for $1 > \lambda > 0$,
\[
\min_{i_1, \ldots, i_m \in \mathbb{Z}^n} |i_1 + i_2 + \cdots + i_m + \lambda| = \lambda.
\]
For $-mn < \lambda < 0$, there exist some positive integers $i'_1, i'_2, \ldots, i'_m$ and $i''_1, i''_2, \ldots, i''_m$ such that
\[
i'_1 + i'_2 + \cdots + i'_m = -[\lambda] \text{ and } i''_1 + i''_2 + \cdots + i''_m = -[\lambda] - 1,
\]
and then,
and hence,
\[ \min_{i_1, \ldots, i_m \in \mathbb{I}_n} |i_1 + i_2 + \cdots + i_m + \lambda| = \min \left\{ \lambda - \lfloor \lambda \rfloor, \lambda - (-\lfloor \lambda \rfloor - 1) \right\}. \]

For \( \lambda < -mn \), we also have,
\[ \min_{i_1, \ldots, i_m \in \mathbb{I}_n} |i_1 + i_2 + \cdots + i_m + \lambda| = |mn + \lambda| = -mn - \lambda. \]

Therefore, we have for \( \lambda \in \mathbb{R} \setminus \mathbb{Z}^- \) with \( \lambda < 1 \),
\[ \frac{1}{|i_1 + i_2 + \cdots + i_m + \lambda|} \leq N(\lambda) = \begin{cases} \frac{1}{\lambda}, & 1 > \lambda > 0; \\ \frac{1}{\min(\lambda - \lfloor \lambda \rfloor, 1 + [\lambda] - \lambda)}, & -mn < \lambda < 0; \\ \frac{1}{-mn - \lambda}, & \lambda < -mn \end{cases}. \]

Then, for all nonzero vector \( x \in \mathbb{R}^n \), we have
\[ |\mathcal{H}_x^m x^m| = \left| \sum_{i_1, i_2, \ldots, i_m = 0}^{n} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m + \lambda} \right| \leq \sum_{i_1, i_2, \ldots, i_m = 0}^{n} \frac{|x_{i_1} x_{i_2} \cdots x_{i_m}|}{|i_1 + i_2 + \cdots + i_m + \lambda|} \leq N(\lambda) \sum_{i_1, i_2, \ldots, i_m = 0}^{n} |x_{i_1}||x_{i_2}| \cdots |x_{i_m}| = N(\lambda) (\sum_{i = 0}^{n} |x_i|)^m = N(\lambda) \|x\|_1^n. \]

For each \( Z_1 \)-eigenvalue \( \mu \) of \( \mathcal{H}_\lambda^m \) with eigenvector \( x \), from (3.2) and \( \|x\|_1 \leq \sqrt{n} \|x\|_2 \), it follows that
\[ |\mu| \left( \frac{1}{n} \|x\|_1^2 \right) \leq |\mu| \|x\|_2^2 = |\mathcal{H}_x^m x^m| \leq N(\lambda) \|x\|_1^m, \]
and hence,
\[ |\mu| \leq nN(\lambda). \]

This completes the proof. \( \square \)

When \( \lambda = 1 \), the following conclusion of Hilbert tensor is easily obtained. Also see Song and Qi [25] for the conclusions about H-eigenvalue and Z-eigenvalue of such a tensor.

**Corollary 3.2.** Let \( \mathcal{H} \) be an \( m \)-th order \( n \)-dimensional Hilbert tensor. Then for all \( Z_1 \)-eigenvalue \( \mu \) of \( \mathcal{H} \),
\[ |\mu| \leq n \sin \frac{\pi}{n}. \]

**Theorem 3.3.** Let \( \mathcal{H}_\infty^m \) be an \( m \)-th order infinite dimensional generalized Hilbert tensor. Assume \( \lambda > 0 \), then for \( Z_1 \)-eigenvalue \( \mu \) of \( \mathcal{H}_\infty^m \),
\[ |\mu| \leq M(\lambda) = \begin{cases} \frac{x}{\sin \lambda \pi}, & 0 < \lambda \leq \frac{1}{2}; \\ \pi, & \lambda > \frac{1}{2}. \end{cases} \]
Proof. For $x \in l^2$, it follows from Lemma 2.2 that

$$|\langle x, H^\infty_{\lambda} x^{m-1} \rangle| = |H^\infty_{\lambda} x^m| = \left| \sum_{i_1, i_2, \ldots, i_m = 0}^{+\infty} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{i_1 + i_2 + \cdots + i_m + \lambda} \right| \leq \sum_{i_1, \ldots, i_m = 0}^{+\infty} \frac{|x_{i_1}||x_{i_2}| \cdots |x_{i_m}|}{i_1 + i_2 + \lambda}$$

$$= \left( \sum_{i_1 = 0}^{+\infty} \frac{|x_{i_1}|}{i_1 + \lambda} \right) \left( \sum_{i_2 = 0}^{+\infty} \frac{|x_{i_2}|}{i_2 + \lambda} \right) \cdots \left( \sum_{i_m = 0}^{+\infty} \frac{|x_{i_m}|}{i_m + \lambda} \right) \leq M(\lambda) \|x\|_1^2 \|x\|_1^{m-2},$$

and so,

$$|\langle x, T_\infty x \rangle| = |\langle x, \|x\|_1^{2-m} H^\infty_{\lambda} x^{m-1} \rangle| = \|x\|_1^{2-m} |H^\infty_{\lambda} x^m| \leq M(\lambda) \|x\|_1^2. \quad (3.4)$$

For each $Z_1$-eigenvalue $\mu$ of $H^\infty_{\lambda}$, there exists a nonzero vector $x \in l^2$ such that

$$T_\infty x = \|x\|_1^{2-m} H^\infty_{\lambda} x^{m-1} = \mu x,$$

and so,

$$\mu \|x\|_1^2 = \mu \langle x, x \rangle = \langle x, \|x\|_1^{2-m} H^\infty_{\lambda} x^{m-1} \rangle = \|x\|_1^{2-m} |H^\infty_{\lambda} x^m|.$$

Therefore, we have

$$|\mu| \|x\|_1^2 = \|x\|_1^{2-m} |H^\infty_{\lambda} x^m| \leq M(\lambda) \|x\|_1^2,$$

and then,

$$|\mu| \leq M(\lambda).$$

This completes the proof. \qed

When $\lambda = 1$, the following conclusion of infinite dimensional Hilbert tensor is easily obtained.

**Corollary 3.4.** Let $H^\infty_{\lambda}$ be an $m$th-order infinite dimensional Hilbert tensor. Then for all $Z_1$-eigenvalue $\mu$ of $H^\infty_{\lambda}$,

$$|\mu| \leq \pi.$$

**Remark 3.1.**

(i) In Theorem 3.1, the upper bound of $Z_1$-eigenvalue of $H^\infty_{\lambda}$ are showed. However the upper bound may not be the best. Then which number is its best upper bounds?

(ii) In Theorem 3.3, the upper bound of $Z_1$-eigenvalue of $H^\infty_{\lambda}$ are showed for $\lambda > 0$, then for $\lambda < 0$ with $\lambda \in \mathbb{R} \setminus \mathbb{Z}^+$, it is unknown whether have similar conclusions or not. And it is not clear whether the upper bound may be attained or cannot be attained.
References


