Invariant manifold growth formula in cylindrical coordinates and its application for magnetically confined fusion

WENYIN WEI (魏文崟)\(^1,2\)
and YUNFENG LIANG (梁云峰)\(^1,3\)

\(^1\) Institute of Plasma Physics, Hefei Institutes of Physical Science, Chinese Academy of Sciences, Hefei 230031, People’s Republic of China
\(^2\) University of Science and Technology of China, Hefei 230026, People’s Republic of China
\(^3\) Forschungszentrum Jülich GmbH, Institut für Energie- und Klimaforschung - Plasmaphysik, 52425 Jülich, Germany

E-mail: y.liang@fz-juelich.de

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Abstract

A spiral ribbon-like pattern of heat deposition has been reported and investigated in the past, of which the field structure is essential to study how to expand the wet area of heat flux to reduce the tolerance-to-heat demand for materials at the divertor in a magnetically confined fusion device. The relevant notions concerning magnetic topology are formalized in this paper to utilize knowledge from dynamical system research. Of great importance to comprehending the topology of general 3D vector fields are cycles and the invariant manifolds grown from saddle cycles, which are analyzed in detail. How the Jacobian of Poincaré map evolves along a cycle is presented. Grown in the directions of the Jacobian eigenvectors at the beginning, the invariant manifolds of saddle cycles are essential to determine the chaotic field regions, which induce a mixing effect inside the plasma. With regard to three-dimensional continuous-time dynamical systems, the governing equation of invariant manifolds in cylindrical coordinates is deduced.

Keywords: magnetic topology, tokamak, invariant manifold

1. Introduction

In the tokamak community, the magnetic topology is, most of the time, assumed to be nested flux surfaces, e.g. in Grad-Shafranov equation, EFIT, VMEC, etc. Based on this assumption, people have established a dedicated theory \([1]\) of flux coordinates in which all magnetic field lines on a flux surface are straight. Nevertheless, the three-dimensional (3D) effect is ubiquitous in real-world fusion experiments since any facility in fusion machines, except the central solenoid and poloidal field coils, is 3D. For example, the toroidal field coils exhibit a ripple effect, the microwave and radio-frequency wave heating impose their distinctive localized distribution pattern of the induced current, and the neutral beam injection (NBI) has an obvious non-axisymmetric current effect. Furthermore, most instability modes of plasma, such as the tearing mode and sawtooth mode, imply a significant 3D topology change of the magnetic field, because they are impossible in an axisymmetric magnetic field. The 3D effect is unavoidable in the magnetically confined fusion research, thereby requiring a deeper comprehension on the global field structure.
In traditional research of magnetic topology, the behaviour of field lines in the stochastic field of a 3D vector field has drawn a lot of attention [2], though the adjective *stochastic* is used in mathematics to depict the phenomena that are of true randomness. The field lines in these regions indeed obey the rules of chaos, i.e. they are intrinsically unpredictable (though deterministic) in the long term, for which we propose to call them *chaotic field regions*. This implies the appearance of island chains when an axisymmetric 3D field is imposed an asymmetric perturbation. The chaos phenomenon emerges from the domain of celestial mechanics. Since the bodies in the Solar System attract each other according to Newton’s law, their motion shall be deterministic. However, even a three-body problem is unpredictable in the long term due to its chaotic nature, let alone the \( n \)-body problem. For the relevant Poincaré maps, the nested invariant 1D tori \( T^1 \) are found near the elliptic fixed points. The nested invariant tori could be destroyed under perturbation. A natural problem is what happens to the dynamical system after the perturbation. Poincaré-Birkhoff theorem addresses this problem, roughly speaking: a homeomorphism which maps an annulus to itself and rotates the two boundaries in opposite directions has at least two fixed points.

The persistence of nested tori under small perturbation is dealt by Kolmogorov-Arnold-Moser (KAM) theorem, which elucidates that the destruction of nested invariant manifolds diffeomorphic to \( n \)-dimensional tori \( T^n = S^1 \times \cdots \times S^1 \) of an \( n \) degrees of freedom completely integrable Hamiltonian system under perturbation is gradual. In other words, as the strength of the perturbation is decreased, the union of the surviving manifolds saturates the whole state space. Poincaré-Birkhoff and KAM theorems jointly determine that the invariant manifolds are destructed under perturbation, and the subprime invariant manifold structure appears recurrently as the perturbation is intensified.

An enormous amount of fusion research has attempted to stimulate a chaotic field layer at the plasma boundary by either resonant magnetic perturbation (RMP) coils [3, 4] or other means to mitigate destructive type-I edge localized modes (ELMs). The theoretical basis was established in 1983 by Cary and Littlejohn [5] to estimate how wide the island chains are when an axisymmetric magnetic field is perturbed by a non-axisymmetric one, after which hundreds of researchers implement RMP coils to mitigate and suppress ELMs [6, 7]. The method is called magnetic spectrum analysis nowadays, heavily relying on Fourier transform of the radial component of the perturbation field, which is a linear operation in functional spaces. Therefore the utility of magnetic spectrum analysis is limited inside the plasma and becomes less and less accurate as the perturbation is strengthened, because Fourier transform is just a linear operation and not capable of explaining the nonlinear behaviour.

With the aid of modern dynamical system theory, the structure of a 3D vector field can be comprehended and analyzed in terms of invariant manifolds [8, 9]. Various numerical methods have been developed to grow them [10–14]. Kuznetsov and Meijer systematically investigated the bifurcation behaviour of 1D and 2D maps when their parameters change [15, 16], presenting diverse analytic and numerical methods to study maps. However, the methods taken by mathematicians are too general to capture the essence of a 3D vector field, leaving the analysis as complicated as before.

Since the magnetic field dominates the plasma transport in magnetically confined fusion machines, the evidence of transversely intersecting invariant manifolds shall be easy to observe, which is a signature of chaos. The invariant manifolds are essential to determine the chaotic field regions, which induce a mixing effect inside the plasma. The plasma edge is not suitable to be characterized by a single closed surface when the 3D effect is strong, for which we propose to use the notion of *invariant manifolds of outmost saddle cycle(s)*. In fact, it has been observed in some existing simulation [17–20] and experiment results. On EAST, the helical current filaments
induced by the lower hybrid wave heating impacted the plasma edge topology and caused an evident splitting of strike points in experiments [21].

If RMP coils are imposed to suppress ELMs in tokamaks, the heat flux pattern at the divertor exhibits a complex toroidally asymmetric distribution, which poses challenges to ITER and DEMO divertor designs [22]. Simulation results demonstrate that the divertor plasma regions with connection to the bulk plasma are dragged further outside when the asymmetry is intensified [23,24]. Generally, the field line connection length and the magnetic footprint (how deep the field lines penetrate into the bulk plasma) distribution at the divertor are apparently ribbon-like.

The toroidally asymmetric divertor heat flux has both advantages and disadvantages. It is certainly desirable to expand the wet regions at the divertor to lower the requirement for the material heat conductivity. The resonant magnetic perturbation does mitigate or even suppress the ELMs, which could cause intolerable transient particle and heat flux. On the other hand, significant heat fluxes may arise far from the strike point originally designed for axisymmetric cases [25], possibly damaging the fragile parts of the divertor. Moreover, it remains unknown whether the total heat flux leaked from the bulk plasma is reinforced by the perturbation.

We are not the first ones to draw the two transversely intersecting manifolds of the hyperbolic cycle(s) for 3D toroidal vector fields. Ottino from the community of fluid mechanics [26], Roeder, Rapoport, and Evans from the fusion community [27,28] have already drawn up the relevant figures. Abdullaev has deduced an approximate (to first order in $\epsilon$ because Poincaré integral is used) analytic implicit expression of the invariant manifolds of the outmost X-cycle when a single-null configuration is perturbed [29]. In our paper, we carry forward the research and directly deduce the intrinsic analytic formula of the invariant manifolds of hyperbolic cycles without need for approximation.

Sect. 2 explains the definitions and denotations used in this paper. The theory core of this paper is put in Sect. 3 From field line tracing to invariant manifolds and Appendix A Proofs, which might be hard to understand at the first time reading for most readers. If one finds it hard to continue reading or follow the proofs, please move to Sect. 4 Demonstration of cycles and invariant manifolds for intuitive examples. Appendix B gives some comparisons with others’ works.

2. Definitions and denotations explained

Motivated by the development of ordinary differential equations, the dynamical system research has harvested enormous useful results [30–33]. Basics of dynamical system and chaos theory are presented concisely in this section to familiarize readers with the terminologies used in this paper. Those readers having sufficient knowledge on math can just have a glance at this section.

2.1 Dynamical system

A discrete-time dynamical system (a.k.a. map) can be thought of as a tuple $(M, f)$, where $f : M \to M$ is a homeomorphism (a one-to-one function that both the function and its inverse are continuous) on a metric space or a differentiable manifold $M$. If the function $f$ instead serves as the rate of state change in $M$, i.e. $\dot{x} = f(x)$, $f : M \to \mathbb{R}^n$, then a continuous-time dynamical system is induced by the given system of ordinary differential equations (ODEs). The solution of the system is called flow, often denoted by $\phi(x_0, t)$ or $\phi^t(x_0)$. The symbol $\phi$ has already been used to denote the azimuthal angle in this paper. How the solution is denoted will be explained later in this subsection. By intersecting trajectories in the state space with an appropriate cross-section (not necessarily an infinite plane), one can acquire a map from a flow. This method is called Poincaré map, the smoothness of which depends on the
smoothness of the inducing field. The set of all $C^r$ vector fields on $M$ is usually denoted by $\mathcal{X}^r(M)$. Fields in this paper are presumed to be at least once continuously differentiable, i.e. of class $C^1$, to avoid strange functions. Although a field is denoted by $B$, it is worth emphasizing that they do not need to be divergence-free in this paper.

A dynamical system is said to be autonomous if its $f$ is independent of the evolution parameter $t$, otherwise we have to rewrite $f$ as $f: (G, M) \to M$, where $G$ is a semigroup such as $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{Z}^+$. To distinguish the continuous dynamics from the discrete one, different terminologies are used for them in this paper, e.g. the terminologies equilibrium point, trajectory and cycle are for continuous-time dynamical systems, while fixed point, orbit and periodic orbit are for those discrete ones.

At the core of this paper are those autonomous dynamical systems $(M, B)$ induced by 3D physical vector fields, where $M \subset \mathbb{R}^3$ and $B \in \mathcal{X}^1(M)$. The solution of such a system $\dot{x} = B(x)$ is denoted by $X(x_0, t)$ in this paper, which initiates from the point $x_0 = (x_{0x}, x_{0y}, x_{0z})$, i.e. $X(x_0, 0) = x_0$. The differentiation symbol $\mathcal{D}$ is defined to be with respect to $x_0$, that is $\mathcal{D}X(x_0, t) := \partial X(x_0, t)/\partial (x_{0x}, x_{0y}, x_{0z})$. In cylindrical coordinates, the corresponding ODE system is generally expressed as below,

\[
\begin{align*}
dx_R/dt &= B_R \\
dx_Z/dt &= B_Z \\
dx_\phi/dt &= B_\phi/R
\end{align*}
\]

where $B_R = B \cdot \hat{e}_R$, $B_Z = B \cdot \hat{e}_Z$, $B_\phi = B \cdot \hat{e}_\phi$, and $(\hat{e}_R, \hat{e}_Z, \hat{e}_\phi)$ is the standard orthogonal basis of cylindrical coordinates. The transformation between the Cartesian coordinate system and the cylindrical one is standard, as shown below,

\[
\begin{align*}
x &= R \cos \phi \\
y &= R \sin \phi \\
z &= Z
\end{align*}
\]

One can change the evolution parameter from $t$ to the azimuthal angle $\phi$,

\[
\begin{align*}
dx_R/dx_\phi &= RB_R/B_\phi \\
dx_Z/dx_\phi &= RB_Z/B_\phi
\end{align*}
\]

The solution of this system is denoted by $X_{pol}(x_0, t)$, which initiates from the point $x_0 = (x_{0R}, x_{0Z})$, i.e. $X_{pol}(x_0, 0) = x_0$. Similarly, $\mathcal{D}X_{pol}(x_0, t) := \partial X_{pol}(x_0, t)/\partial (x_{0R}, x_{0Z})$. Sometimes, we set a non-zero starting angle and denote the solution by $X_{pol}(x_0, \phi_0, \phi_2)$, i.e. $X_{pol}(x_0, \phi_0, \phi_2)$ starts at $\phi_2$ from $x_0$, and reaches the ending section $\phi_e$. The argument $x_0$ is often omitted for notational convenience, e.g. $\mathcal{D}X_{pol}(\phi_0, \phi_2)$ is short for $\mathcal{D}X_{pol}(x_0, \phi_0, \phi_2)$. Differentiating both sides of Eq. (3) w.r.t. $x_0$ immediately lets one obtain the governing equation of $\mathcal{D}X_{pol}(x_0, \phi_0, \phi_2)$,

\[
\frac{\partial}{\partial \phi_e} X_{pol}(x_0, \phi_0, \phi_2) = \frac{\partial RB_{pol}(x_0, \phi_0, \phi_2)}{\partial \phi} (X_{pol}(x_0, \phi_0, \phi_2), \phi_e)
\]

(3 revised)

\[
\frac{\partial}{\partial \phi_e} \mathcal{D}X_{pol}(x_0, \phi_0, \phi_2) = \frac{\partial RB_{pol}(x_0, \phi_0, \phi_2)}{\partial (R, Z)} (X_{pol}(x_0, \phi_0, \phi_2), \phi_e) \mathcal{D}X_{pol}(x_0, \phi_0, \phi_2),
\]

(4)

which can be written more compactly with $x_0$ omitted and $A(\phi) := \frac{\partial RB_{pol}(x_0, \phi_0, \phi_2)}{\partial (R, Z)} (X_{pol}(x_0, \phi_0, \phi_2), \phi)$,

\[
\frac{\partial}{\partial \phi_e} \mathcal{D}X_{pol}(\phi_0, \phi_2) = A(\phi_e) \mathcal{D}X_{pol}(\phi_0, \phi_2),
\]

(4 abbreviated)
the initial condition of which is $D\mathbf{X}_{pol}(\phi_s, \phi_c) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Similarly, for Cartesian coordinates,

$$\frac{\partial}{\partial t} D\mathbf{X}(x_0, t) = \nabla \mathbf{B}(\mathbf{X}(x_0, t)) D\mathbf{X}(x_0, t).$$

(5)

In control theory, the notion of state-transition matrix $\Phi(t, t_0)$ is essentially the same as $D\mathbf{X}_{pol}(\phi_s, \phi_c)$, while people in the ODE community call it the fundamental solution $\mathbf{F}(t, t_0)$, let $\mathbf{A}(t)$ be an $n \times n$ square matrix function. For the linear system $\dot{x} = \mathbf{A}(t)x(t)$, $x(t_0) = x_0 \in \mathbb{R}^n$, its solution can be represented by the state-transition matrix as $x(t) = \Phi(t, t_0)x(t_0)$. The authors prefer the denotation $D\mathbf{X}_{pol}(\phi_s, \phi_c)$ than $\Phi(t, t_0)$, because $D$ highlights the differentiation is w.r.t. $[x_0R, x_0Z]$ rather than the evolution parameter $\phi$. If $\mathbf{A}(t)$ is periodic in $t$, people often call $\Phi(t_0 + T, t_0)$ the monodromy matrix $\mathbf{M}(t)$. $RB_{pol}/B_\phi(R, Z, \phi)$ is of course $2\pi$-periodic in $\phi$. The Poincaré map at R-Z section $\phi \in \mathbb{R}$, denoted by $\mathcal{P}(x_0, \phi)$, satisfies

$$\mathcal{P}(x_0, \phi) = \begin{cases} \mathbf{X}_{pol}(x_0, \phi, \phi + 2\pi) & \text{if } B_\phi \text{ is positive everywhere}, \\ \mathbf{X}_{pol}(x_0, \phi, \phi - 2\pi) & \text{if } B_\phi \text{ is negative everywhere}. \end{cases}$$

(6)

If $x_0$ is on a cycle of $m$ toroidal turn(s), i.e. $\mathbf{X}_{pol}(x_0, \phi, \phi + 2m\pi) = x_0$, then naturally $RB_{pol}/B_\phi(\mathbf{X}_{pol}(x_0, \phi), \phi)$ is $2m\pi$-periodic in $\phi$. In our paper, $D\mathcal{P}^{\infty}(x_0, \phi)$ serves as the monodromy matrix.¹

2.2 Chaos and invariant manifolds

The chaos phenomenon is famous for that it makes the long-term prediction of a nonlinear dynamical system unreliable. Even the simple 3D FLT ODE system (3) can be chaotic. An orbit of its Poincaré map may or may not be limited in a bounded region, i.e. the corresponding chaotic region may not or may extend to the infinity. In both cases, the field line tracing error caused by the uncertainty of the initial position accumulates exponentially with respect to the evolution time. Although this error has a supremum in the former case (since the region is bounded), it still makes accurate prediction difficult. Such sensitivity of initial condition is intrinsic for chaotic dynamical systems, regardless of how precise the adopted numeric method is.

Though complicated, the chaotic regions of a map $f$ have their distinctive disciplines, albeit not obvious for the uninitiated. An invariant set $C \subset M$ under $f$ means that once an orbit of $f$ enters the set $C$, it can never go out of $C$. For a discrete-time dynamical system $(M, f)$, the stable and unstable manifolds for a hyperbolic ² invariant set $S$, under a homeomorphism $f$, are defined by

$$W^s(S) := \{ p \in M \mid \omega(p) = S \},$$

$$W^u(S) := \{ p \in M \mid \alpha(p) = S \},$$

(7)

(8)

where $\omega$- and $\alpha$-limit sets of a point $p$ are defined to be the sets of those points $y$ for which there exists a strictly increasing sequence of natural numbers $\{n_k\}$ such that $f^{n_k}(p) \to y$ and

¹Higher order derivatives of $\mathbf{X}_{pol}(x_0, \phi_s, \phi_c)$ can be denoted by $D^k\mathbf{X}_{pol}(\phi_s, \phi_c)$, $k \geq 1$. Similarly, higher order derivatives of $\mathcal{P}^{\infty}(x_0, \phi)$ can be denoted by $D^k\mathcal{P}^{\infty}(\phi)$.

²The adjective hyperbolic is to simplify the definitions of stable and unstable manifolds. For a partially hyperbolic invariant set, the definitions should additionally require the orbits converge at an exponential rate. On center manifolds, orbits can approach (or get away from) an invariant set, but they are not allowed to do so at an exponential rate. Most of our readers might be uninterested in the concepts. For a definition of hyperbolic for a map, we refer to [34], Definition 4.1, Chap. 4 Hyperbolic Sets. For a definition of hyperbolic for a map, we refer to [35], Sect. 9.2. Stable and unstable manifolds, where the definition of convergence with an exponential rate is also discussed around Eq. (9.9), p. 257. The terminology hyperbolic from the ODE community is not well-chosen, which does not mean saddle. Many people have complained about this. As said by S. H. Strogatz, “...the fixed point is often called hyperbolic (This is an unfortunate name—it sounds like it should mean ‘saddle point’—but it has become standard.)” [36] p. 155.
$f^{-n_k}(p) \to y$ as $k \to +\infty$, respectively. (see [34] p. 2) When the map is unclear, $f$ is added to the argument list, like $W^u(f, p)$ and $\omega(f, p)$. Sometimes, $S$ is nicknamed past and future for unstable and stable manifolds, respectively.

These things are defined similarly for continuous dynamics. For example, $\omega$- and $\alpha$-limit sets of a point $p$ are defined to be the sets of those points $y$ for which there exists a strictly increasing sequence of positive numbers $\{t_k\}$ such that $X(p, t_k) \to y$ and $X(p, -t_k) \to y$ as $k \to +\infty$, respectively. Moreover, let $\gamma$ be a cycle of an autonomous $n$-dim system $(M, B)$ and $p$ a point on $\gamma$. Conventionally, $\gamma$ is said to be a hyperbolic cycle of $B$ if $p$ is a hyperbolic fixed point of the Poincaré map $P : V \subset \Sigma \to \Sigma$, where $\Sigma$ is a section transversal to $B$ through $p$, and $V$ is a neighborhood of $p$ in $\Sigma$; see [37] p. 95. Let $n = 3$, and $\gamma$ be a cycle of $m$ toroidal turns in cylindrical coordinates. Then, the 2D manifold $W^{u,s}(B, \gamma)$ consists of all the corresponding 1D manifolds $W^{u,s}(P^n(\phi), x(\phi))$, under the Poincaré maps $P^n(\phi)$ at all sections $\phi$, i.e.

$$W^{u,s}(B, \gamma) = \bigcup_{\phi \in [0, 2\pi]} \{x(R, Z, \phi) \in M | (R, Z) \in W^{u,s}(P^n(\phi), x(\phi))\},$$

where $x(\phi)$ denotes the $(R, Z)$ coordinates of the cycle $\gamma$ at $\phi$ angle, and $P(\phi)$ denotes the Poincaré map from the section $\phi$ itself under $B$. Our Poincaré sections in this paper are chosen to be the semi-infinite planes $\Sigma(\phi) = \{x(R, Z, \phi) \in M | (R, Z) \in \mathbb{R}^+ \times \mathbb{R}\}$. If a trajectory flies around $\Sigma$ through the $R < 0$ semi-infinite plane, then by definition our Poincaré map does not record its corresponding striking point at $R < 0$.

The fundamental No-Intersection theorem [38] says that two distinct trajectories cannot intersect (in a finite period of time). Nor can a single trajectory cross itself at a later time. Therefore, an invariant manifold also can not cross itself. Given two different invariant sets $S_1$ and $S_2$, their unstable manifolds $W^u(S_1)$ must be disjoint, i.e. $W^u(S_1) \cap W^u(S_2) = \emptyset$, because the points on these two manifolds have distinct pasts. The stable manifolds are similar. The transverse intersection of invariant manifolds is only possible between a stable one $W^s(S_1)$ and an unstable one $W^u(S_2)$. The stable and unstable manifolds of the distinct $S_1$ and $S_2$ may intersect transversely into heteroclinic trajectories/orbits (trajectories for flows, orbits for maps). The intersection is thereby called a heteroclinic intersection. If $S_1 = S_2$, one call it a homoclinic intersection. On the other hand, if $S_1 \neq S_2$, and a branch of $W^s(S_1)$ is identical with a branch of $W^u(S_2)$, then we call the overlapped branch a heteroclinic connection between $S_1$ and $S_2$. By the term branch, we mean a disconnected component of an invariant manifold. For example, a saddle fixed point of a 2D map has two eigenvectors and four invariant branches. If $S_1 = S_2 = S$, we call the branch a homoclinic connection, nicknamed self-connection of $S$.

The single-null topology of an ideally axisymmetric tokamak is a good example to illustrate the concepts aforementioned. Denote its X-cycle by $\gamma$. Then, one of the two stable branches of $W^u(\gamma)$ completely overlaps with one of the two unstable branches of $W^u(\gamma)$, serving as the last closed flux surface (LCFS). The LCFS is indeed a self-connection of $\gamma$. The double-null topology is more complicated. Let the upper and lower $X$-cycles be denoted by $\gamma_{up}$ and $\gamma_{low}$, respectively. If the two downward invariant branches of $\gamma_{up}$ perfectly overlap with the two upward branches of $\gamma_{low}$, then we say there exist two connections between $\gamma_{up}$ and $\gamma_{low}$. But usually it is not the case. The two downward branches of $\gamma_{up}$ do not overlap with the two upward branches of $\gamma_{low}$, which is called a disconnected double-null configuration, as shown in Fig. 4, Sect.4.3.

3. **From field line tracing to invariant manifolds**

$DX$ and $DX_{pol}$ implies the change of differential volume and area during FLT, respectively. Suppose a 2D map is written as $(x, y) \mapsto (u, v)$. The differential area expands, shrinks, or keeps
constant after being mapped, as revealed by the following exterior product of differential 1-forms,
\[ du \wedge dv = \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) \wedge \left( \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) = \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) dx \wedge dy. \tag{10} \]

As \( X_{pol}(\phi_s, \phi_e) \) is a typical 2D map from the section \( \phi_s \) to \( \phi_e \), the determinant of \( DX_{pol}(\phi_s, \phi_e) \), denoted by \( |DX_{pol}(\phi_s, \phi_e)| \), is indeed the same thing as \( \partial_x u \partial_y v - \partial_y u \partial_x v \). One could be curious about the geometric meaning of \( |DX_{pol}(\phi_s, \phi_e)| \) and conjecture that it must be related to the divergence of the field, since people have known that the flux enclosed by a magnetic flux tube is conserved, \( i.e. \ B_\theta(X_{pol}(\phi_s, \phi_e), \phi_e) du \wedge dv = B_\theta(X_{pol}(\phi_s, \phi_e), \phi_e) dx \wedge dy \). To be more accurate,
\[ |DX(x_0, t)| = e^{\int_0^t \nabla B(X(x_0, t)) dt} = e^{\int_0^t \nabla B(X(x_0, t)) dt}, \tag{11} \]
which indicates that, for a divergence-free field, \( |DX(x_0, t)| \) is always zero. This equation has been well-known in the ODE community. With regard to a general field which is not almost everywhere divergence-free, how to solve for \( |DX_{pol}(\phi_s, \phi_e)| \)? The following formula is deduced (proof in Appendix A.1) to reveal the relationship between \( |DX_{pol}(\phi_s, \phi_e)| \) and the divergence along the corresponding trajectory \( X_{pol}(\phi_s, \phi_e) \), \( \phi_s \leq \phi \leq \phi_e \).
\[ |DX_{pol}(\phi_s, \phi_e)| = \exp \left( \int_{\phi_s}^{\phi_e} \frac{R(\nabla \cdot B)}{B_\phi} d\phi \right) \frac{B_\theta|_{\phi_e}}{B_\theta|_{\phi_s}}, \tag{12} \]
which applies to all 3D physical fields, no matter whether they are divergence-free or not.

More importantly, \( D^{\pm m}(x_0, \phi) \) with \( x_0 \) on an X-cycle \( \gamma \) of \( m \) toroidal turn(s) decides the two X-leg directions of the X-point. \( D^{\pm m}(x_0, \phi) = DX_{pol}(x_0, \phi + 2m\pi) \) if \( B_\theta \) is positive everywhere. It is the two eigenvectors of \( D^{\pm m}(x_0, \phi) \) that dictate towards which direction the two invariant manifolds of that X-cycle grow at the beginning. However, \( D^{\pm m}(x_0, \phi) \) itself is more difficult to solve for than its determinant, which is a constant independent of \( \phi \) for the cycle. This is because the right-hand side of Eq. (12) becomes a constant with \( \phi_e = \phi_s + 2m\pi \), \( i.e. \)
\[ |DX_{pol}(\phi_s, \phi_s + 2m\pi)| = \exp \left( \int_{\phi_s}^{\phi_s+2m\pi} \frac{R(\nabla \cdot B)}{B_\phi} d\phi \right) \frac{B_\theta|_{\phi_s+2m\pi}}{B_\theta|_{\phi_s}} = \exp \left( \int_{\phi_s}^{\phi_s+2m\pi} \frac{R(\nabla \cdot B)}{B_\phi} d\phi \right). \tag{13} \]
Unlike the determinant \( |D^{\pm m}(x_0, \phi)| \), the matrix \( D^{\pm m}(x_0, \phi) \) w.r.t. the azimuthal angle \( \phi \). The evolution rule of \( D^{\pm m}(x_0, \phi) \) is revealed in section 3.1.

To calculate \( DX_{pol}(\phi_s, \phi_e) \) by integrating Eq. (4) requires that \( B_\theta \) on the orbit do not change its sign. Otherwise, \( RB_\theta \) would be undefined due to zero \( B_\theta \). These orbits are not useless and could be well-defined in Cartesian coordinates. Suppose an orbit goes from the R-Z section \( \phi_s \) to the other one \( \phi_e \), during which \( B_\theta \) may change its sign for several times. The zero \( B_\theta \) singularities cause inconvenience to solving for the \( DX_{pol}(\phi_s, \phi_e) \). To get around the zero \( B_\theta \) singularity issue, one can solve for the corresponding \( DX \) first and then change its coordinates back to the cylindrical system. The following \( DX \) to \( DX_{pol} \) formula (proof in Appendix A.2) tells how to do so,
\[ \begin{bmatrix} 1 & -R R_2 \end{bmatrix} \begin{bmatrix} \cos \phi & -R \sin \phi \\ \sin \phi & R \cos \phi \end{bmatrix}^{-1} DX \begin{bmatrix} \cos \phi & -R \sin \phi \\ \sin \phi & R \cos \phi \end{bmatrix}_{\text{start}} = DX_{pol} \begin{bmatrix} * \end{bmatrix}, \tag{14} \]
where the subscripts \( \text{start} \) and \( \text{end} \) mean that the corresponding matrices are evaluated at the starting and ending point of this orbit, respectively, \( i.e. \ (X_{pol}(\phi_s, \phi_s), \phi_s) \) and \( (X_{pol}(\phi_s, \phi_e), \phi_e) \).
3.1 The evolution of $\mathcal{DP}^{\pm m}$ along a cycle

For a cycle of $m$ toroidal turn(s), there must exist some relationship among the $\mathcal{DP}^{\pm m}(\phi)$ matrices at neighboring $\phi$ sections, without which the calculation of their eigenvectors would spend unnecessarily enormous computational resources. The most primitive approach is definitely repeating the integration of Eq. (4) from $\phi_i$ to $\phi_i + 2m\pi$ once and once again for various $\phi_i$, i.e. from $\phi_1$ to $\phi_1 + 2m\pi$, from $\phi_2$ to $\phi_2 + 2m\pi$... To avoid this horribly primitive approach, we deduced the following $\mathcal{DP}^{\pm m}$ evolution formula \(^3\) (proof in Appendix A.3) to unearth how $\mathcal{DP}^{\pm m}(\phi)$ varies along the cycle,

$$\frac{d}{d\phi} \mathcal{DP}^{\pm m}(\phi) = [A(\phi), \mathcal{DP}^{\pm m}(\phi)],$$

where the square bracket denotes the commutator, i.e. $[A, B] = AB - BA$. In the dummy X-cycle demonstration at Fig. 2(a, b), Sect. 4.2, we draw arrows to show the directions of eigenvectors.

Furthermore, it is desirable to deduce how an eigenvector of $\mathcal{DP}^{\pm m}(\phi)$ evolves along an X-cycle, so that one can get rid of the arbitrariness of the computed eigenvector direction, which depends on the specific eigen-decomposition numeric algorithm. If the eigenvector rotates a lot during evolution, the employed numerical method might give a reversed direction without consistency, i.e. the eigenvector may jump to the other side suddenly. The following $\mathcal{DP}^{\pm m}$ eigenvector formula (proof in Appendix A.4) extracts the underlying rule governing the evolution of $\mathcal{DP}^{\pm m}(\phi)$ eigenvectors along the cycle. Let the eigenvectors of $\mathcal{DP}^{\pm m}(\phi)$ be denoted by $\psi_i = [\cos \theta_i(\phi), \sin \theta_i(\phi)]^T$, $i \in \{1, 2\}$. The derivative of $\Theta(\phi) := \text{diag}(\theta_1, \theta_2)$ w.r.t. $\phi$ satisfies

$$\Theta' = \begin{pmatrix} -1 & \psi \Lambda - \mathcal{DP}^{\pm m} \begin{pmatrix} -1 & \psi \end{pmatrix} \end{pmatrix}^{-1} (\mathcal{DP}^{\pm m})' \psi,$$

where $\psi := [\psi_1, \psi_2]$ and $\Lambda := \text{diag}(\lambda_1, \lambda_2)$. We discovered in numeric implementation that the formula above, though accurate, but encounters some numeric issue while handling the X-cycles in island chains, because the two eigenvectors are so close to each other that some matrices in this formula might become pretty singular. A much more robust way is to directly use $\mathcal{DP}^{\pm m}(\phi)$ evolution formula (16) to evolve $\mathcal{DP}^{\pm m}(\phi)$ and later comb the directions of eigenvectors.

Traditionally, fixed points of 2D maps are classified into hyperbolic, elliptic, and parabolic types based on their Jacobian eigenvalues. For the cycles of 3D flows, we want to imitate this naming convention. It is worth emphasizing that the eigenvalues of $\mathcal{DP}^{\pm m}(\phi)$ keep constant during evolution.\(^4\) Hence, it is safe to classify a cycle $\gamma$ of $m$ toroidal turn(s) by its $\mathcal{DP}^{\pm m}$ eigenvalues. The $\lambda$-invariance ensures the safety of such classification, because one does not need to worry about that the $\mathcal{DP}^{\pm m}(\phi)$ eigenvalues at different $\phi$ are different. If both eigenvalues of $\mathcal{DP}^{\pm m}$ are

\(^3\)The $\mathcal{DP}^{\pm m}$ evolution formula can be generalized to general cycles in autonomous n-dim flows ($n \geq 2$), not only the 3D ones. For other dimensions than $n = 3$, we prefer the notation $[\mathcal{D}(X(x_0, t)), \mathcal{D}X(x_0, t)]$ than $[\mathcal{D}^{\pm m}(\phi)]$, where $T$ is the period of the cycle, because the former one does not rely on the choice of Poincaré section. Here we give the n-dim version of Eq. (16) in Cartesian coordinates

$$\frac{d}{dt} \mathcal{D}X(x_0, t, T) = [\nabla B(X(x_0, t)), \mathcal{D}X(x_0, t, T)].$$

\(^4\)This is a well-known fact in the ODE community, because $\mathcal{D}(\phi_2)$ and $\mathcal{D}(\phi_1)$ are obviously similar by $D(\phi_2) = DX_{pol}(\phi_2, \phi_1)^{-1} \mathcal{D}(\phi_1) DX_{pol}(\phi_2, \phi_1)$. The commutator form of the right-hand side of this formula also ensures the $\lambda$-invariance. Let $x_i$ be a right eigenvector of $\mathcal{DP}^{\pm m}$ and $y_i$ the corresponding left eigenvector,

$$\mathcal{DP}^{\pm m} x_i = \lambda_i x_i, \quad y_i^T \mathcal{DP}^{\pm m} = \lambda_i y_i^T.$$
not on the unit circle $S$ of $C$, the cycle is said to be hyperbolic. If only one eigenvalue on the unit circle, the cycle is called partially hyperbolic (but not hyperbolic). If both eigenvalues are on the unit cycle of $C$ but neither equal 1 nor $-1$, the cycle is defined to be elliptic. If the two eigenvalues are identical to 1 or $-1$, the cycle is defined to be parabolic.

We further define the saddle cycles to be the ones with $|\lambda_1| < 1$, $|\lambda_2| > 1$. Those saddle cycles with both eigenvalues negative (resp. positive) are called Möbiusian (resp. non-Möbiusian). The cycles with both $\lambda$ inside (resp. outside) the unit circle $S$ of $C$ are defined to be sinking (resp. sourcing) cycles.

\[
\begin{aligned}
\text{hyperbolic} & \quad \text{saddle} & \quad \text{non-Möbiusian} & \quad \text{if both $\lambda$ positive} \\
\text{partially hyperbolic} & \quad \text{sinking} & \quad \text{Möbiusian} & \quad \text{if both $\lambda$ negative} \\
\text{non-hyperbolic} & \quad \text{sourcing} & \quad \text{if both $\lambda$ inside $S$} \\
& & \quad \text{if one eigenvalue $\lambda = 1$ or $-1$, while the other $\lambda \in \mathbb{R} \setminus \{1,-1\}$} \\
& & \quad \text{elliptic if both $\lambda$ on $S$ but $\neq \pm 1$} \\
& & \quad \text{parabolic if both $\lambda = 1$ or $-1$}
\end{aligned}
\]

Magnetic fields are typical divergence-free fields, in which the so-called X-cycles, O-cycles, and the cycles on rational flux surfaces are indeed hyperbolic (meanwhile saddle), elliptic, and parabolic, respectively. In particular, an O-cycle of $m$ toroidal turn(s) with rotational transform $\tau = k/2m$, $k \in \mathbb{Z}$, is parabolic.

### 3.2 Invariant manifold growth formula in cylindrical coordinates

Consider that an invariant manifold of a hyperbolic cycle $\gamma$ may grow endlessly, then one of the two parameters of the manifold is naturally chosen to be the arc length $s$ of the curves intersected by the 2D manifold $W^{u/s}(B, \gamma)$ and R-Z cross-sections. For the other coordinate, the azimuthal angle $\phi$ is chosen. It is defined that $s = 0$ on the cycle and $s$ increases towards the positive infinity as the manifold grows away from the cycle. Our diagram (Fig. 6) in Appendix A.5 could be helpful for readers to understand the geometry, which illustrates the relationship among the differentials used. The diagram is put in Appendix because those readers who follow the proof there would need it more.

An invariant branch of $\gamma$ is parameterized to be $X^{u/s}(s, \phi) = (X_R^{u/s}(s, \phi), X_Z^{u/s}(s, \phi))$, where the superscripts $u$ and $s$ indicate whether the branch is unstable or stable. To deduce the governing equation of $X^{u/s}(s, \phi)$, one simply needs to analyse the differential relationship appearing in FLT, which is concluded in the following invariant manifold growth formula (proof in Appendix A.5, [39]).

\[
\begin{aligned}
\lambda' = y_i^T (D \phi^{\pm u} y_i') x_i \\
= y_i^T (A \phi^{\pm u} - D \phi^{\pm u} A)x_i = y_i^T (A(x_i) - (\lambda_i y_i^T) A)x_i = 0. \\
\end{aligned}
\]

\[
\text{This definition is equivalent with the conventional one mentioned in Sect. 2.2.}
\]
with the initial condition $\partial_s X^{u/s}(s, \phi)\big|_{s=0}$ set to be the normalized eigenvector of $DP^n(\phi)$. Naturally, $\partial_s X^{u/s}(s, \phi)$ is $2\pi$-periodic in $\phi$ for a non-Möbiusian saddle cycle. The denominator is essentially $ds/d\phi$, so the sign $\pm$ shall take $+$ if the field line moves away from the cycle as $\phi$ increases. Otherwise, it takes $-$.

For Möbiusian saddle cycles, the invariant manifolds can be grown similarly with some subtle difference. A non-Möbiusian saddle cycle has two invariant branches for each (un)stable manifold. With regard to a Möbiusian saddle cycle, we consider the two branches for a (un)stable manifold as a whole branch (since they are connected), and double the period of $X^{u/s}(s, \phi)$ in $\phi$ from $2\pi$ to $2 \cdot 2\pi$. $X^{u/s}(s, \phi)$ is opposite $X^{u/s}(s, \phi + 2\pi)$ across the cycle $\gamma$. Then the growth formula works again.

The growth formula would not grow a rational flux surface from a parabolic cycle on that surface, since any field line does not cover that surface. To grow this surface, the $DP^m$ evolution formula is sufficient. One simply needs to move the cycle in the directions of the $DP^m$ eigenvectors step by step. The role of $DP^m$ evolution formula is to accelerate the computation of $DP^m(\phi)$ at all sections. For an irrational flux surface, one can choose a fake “cycle” which does not obey the FLT ODE system (3) and then employ the growth formula. For example, in an axisymmetric field, pick a “cycle” whose $R$, $Z$ coordinates are constants. The growth formula then degenerates into

$$\frac{\partial X^{u/s}}{\partial s} = \left( \frac{RB_{pol}}{B_\phi} - \frac{\partial X^{u/s}}{\partial \phi}\right) \left|_{s=0} \right| \pm \cdots \pm_2,$$

therefore $\partial X^{u/s}/\partial s$ is parallel to $B_{pol}$ and of unit length.

If the only known is limited to one section $\phi$, i.e. the Poincaré map $P(x_0, \phi)$ and its Jacobian $DP(x_0, \phi)$ are unknown except at a given $\phi$ (so $\phi$ will be omitted in this paragraph). Let $\{x_i\}_{i=1}^m$ be a hyperbolic periodic orbit under $P$. Parameterize an 1D invariant branch of $W^{u/s}(P, \{x_i\}_{i=1}^m)$ by its arc length $s$ as $X^{u/s}(s) : \mathbb{R}_\geq \to \mathbb{R}^3$, whose inverse is denoted by $s(X) : \mathbb{R}_\geq \supset X^{u/s}(\mathbb{R}_\geq) \to \mathbb{R}_\geq$. Then one can acquire the following equations to grow the manifold by simple analysis along the branch:

$$\frac{dX^u(s)}{ds} = DP^m\left(P^{-m}\left(X^u(s)\right)\right) \cdot \frac{dX^u}{ds}\left(\left|\frac{d}{ds}\left(P^{-m}\left(X^u(s)\right)\right)\right| \pm \cdots \pm_2\right),$$

$$\frac{dX^s(s)}{ds} = DP^{-m}\left(P^{-m}\left(X^s(s)\right)\right) \cdot \frac{dX^s}{ds}\left(\left|\frac{d}{ds}\left(P^{-m}\left(X^s(s)\right)\right)\right| \pm \cdots \pm_2\right),$$

For a 3D continuous-time dynamical system $(M, B)$, $M \subset \mathbb{R}^3$ and $dx/dt = B(x)$, an invariant manifold $T \subset \mathbb{R}^3$ is called an invariant 2-torus, if there exists a diffeomorphism $\phi : T \to \mathbb{T}^2$ into the standard 2-torus $\mathbb{T}^2 := S \times S$ such that the resulting motion on $\mathbb{T}^2$ is uniform linear, i.e. $d\phi(s)/dt = \omega$ where $\omega = (\omega_1, \omega_2)$ is a constant vector. If the frequency vector $\omega$ of an invariant 2-torus $T$ is rationally dependent, i.e. commensurable, i.e. there exists a vector $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$ such that $k \cdot \omega = 0$, then $T$ is called a rational flux surface, otherwise an irrational one.
where the ellipsis dots \( \cdots \) in the denominators are the same as the numerators, which serve to normalize. The equations above also hold for an invariant circle \( C \) intersected by a flux surface \( T \) and the R-Z section \( \phi \). The only thing that needs to be modified is the circle should be parameterized as a periodic function \( X : \mathbb{R} \rightarrow C \subset \mathbb{R}^2 \), whose inverse \( s(X) : C \rightarrow \mathbb{R} \) is now a multivalued function.

4. Demonstration of cycles and invariant manifolds

Having developed the systematic theory to characterize the invariant manifolds in 3D autonomous flows, we have implemented the formulas to present our readers with a vivid picture. Two analytic and one real-world examples are exhibited in this section. As the simplest model, the first analytic model is a saddle cycle as shown in Fig. 1, which can be either non-Möbiusian or Möbiusian. Next is the more complicated dummy X-cycle model as shown in Fig. 2, which is reformed the first model by twisting the cycle, i.e., let the \( R, Z \) coordinates of the cycle rely on \( \phi \), instead of being constants. In these two analytic examples, we exhibit the technique to construct a field by the expected orbits. At last, a time slice of the magnetic field given by EFIT on EAST is taken as a real-world example as shown in Fig. 3 and Fig. 4, with RMP as the non-axisymmetric factor.

4.1 Non-Möbiusian/Möbiusian saddle cycles

In this section, a saddle cycle model is constructed as an analytic example. Let \( B_\phi := R_0 B_{\phi 0} / R \) to simulate a tokamak, where \( B_{\phi 0} \) denotes the toroidal field magnitude at the axis \( R_0 \). Suppose \( B_\phi \) is positive. An orbit on the invariant manifolds of a Möbiusian cycle is expected to be like

\[
X_{\text{pol}}(\phi) = \begin{cases} 
|\lambda_u|^{\phi/2\pi} \begin{bmatrix} \cos \theta_u(\phi) \\ \sin \theta_u(\phi) \end{bmatrix} + \begin{bmatrix} R_0 \\ Z_0 \end{bmatrix}, & \text{(for unstable trajectories)} \\
|\lambda_s|^{\phi/2\pi} \begin{bmatrix} \cos \theta_s(\phi) \\ \sin \theta_s(\phi) \end{bmatrix} + \begin{bmatrix} R_0 \\ Z_0 \end{bmatrix}, & \text{(for stable trajectories)}
\end{cases}
\]  

(23)

where \( \lambda_u, \lambda_s \) (independent of \( \phi \)) denote the two eigenvalues of \( D P^m \), \( \theta_u(\phi), \theta_s(\phi) \) denote the corresponding two eigenvectors. If both \( \theta_u \) and \( \theta_s \) satisfy \( \theta(\phi + 2\pi) = \theta(\phi) + (2k + 1)\pi, k \in \mathbb{Z} \), then the cycle is Möbiusian; if them satisfy \( \theta(\phi + 2\pi) = \theta(\phi) + 2k\pi, k \in \mathbb{Z} \), then the cycle is non-Möbiusian.

Let \( \Delta X_{\text{pol}}(\phi) := X_{\text{pol}}(\phi) - [R_0, Z_0]^T \). For the unstable trajectories, \( \Delta X'_{\text{pol}}(\phi) \) is

\[
\frac{d}{d\phi} \Delta X_{\text{pol}}(\phi) = \begin{bmatrix} |\lambda_u|^{\phi/2\pi} \left[ \begin{array}{c} \cos \theta_u \\ \sin \theta_u \end{array} \right] \\ |\lambda_u|^{\phi/2\pi} \left[ \begin{array}{c} -\theta'_{u} \\ \sin \theta_u \end{array} \right] \end{bmatrix} + \begin{bmatrix} \cos \theta_u \\ -\theta'_{u} \end{bmatrix} \Delta X_{\text{pol}}(\phi) = \begin{bmatrix} \left( \ln |\lambda_u| \right)_{2\pi m \pi} \\ \left( \ln |\lambda_u| \right)_{2\pi m \pi} \end{bmatrix} \Delta X_{\text{pol}}(\phi) + \begin{bmatrix} \left( \ln |\lambda_u| \right)_{2\pi m \pi} \\ \left( \ln |\lambda_u| \right)_{2\pi m \pi} \end{bmatrix} \Delta X_{\text{pol}}(\phi).
\]  

(24)
The FLT equation turns out to be
\[ \operatorname{Eq. (equal\ at\ θ)} \]
Eigen decomposition is employed to make it possible to let the first orders of the equations above equal at \( θ_u \) and \( θ_s \), respectively. As the cost of such decomposition, we have to let \( θ'_u = θ'_s = θ' \).

[\textbf{Figure 1}: (a, b) show the same Möbiusian saddle cycle, of which the field is constructed by Eq. (30) with parameters: \((R_0, Z_0) = (1, 0, 0), B_{q0} = 2.5, \theta_u(ϕ) = 1/2, \theta_s(ϕ) = 1/2, \lambda_{u/s} = -e^{±1/3}.\) The sprouts of two invariant branches are plotted (red for \( \mathcal{W}^u \), blue for \( \mathcal{W}^s \)). (a) and (b) draw the manifolds on \( φ \in [0, 2π] \) and \([0, 4π]\) for a half and full poloidal turn, respectively. An orbit on the unstable branch is drawn for three toroidal turns, i.e. \( φ \in [0, 6π] \).

It is similar for stable trajectories, with \( λ_u, \theta_u(ϕ) \) replaced by \( λ_s, \theta_s(ϕ) \). Recall the FLT Eq. (3) in cylindrical coordinates is \( ΔX'_\text{pol}(ϕ) = RB_{\text{pol}}/B_ϕ (X_{\text{pol}}(ϕ), ϕ) \). Expand \( RB_{\text{pol}}/B_ϕ \) around the cycle,

\[
\frac{\partial RB_{\text{pol}}/B_ϕ}{\partial (R, Z)} (R, Z, ϕ) = \frac{\partial RB_{\text{pol}}/B_ϕ}{\partial (R, Z)} (ϕ) \Delta X_{\text{pol}}, \quad \text{if } \Delta X_{\text{pol}} \text{ in direction } \theta_u
\]

\[
\frac{\partial RB_{\text{pol}}/B_ϕ}{\partial (R, Z)} (ϕ) = \frac{\partial RB_{\text{pol}}/B_ϕ}{\partial (R, Z)} (ϕ) \Delta X_{\text{pol}}, \quad \text{if } \Delta X_{\text{pol}} \text{ in direction } \theta_s
\]

Eigen decomposition is employed to make it possible to let the first orders of the equations above equal at \( θ_u \) and \( θ_s \), respectively. As the cost of such decomposition, we have to let \( θ'_u = θ'_s = θ' \).

\[
\frac{\partial RB_{\text{pol}}/B_ϕ}{\partial (R, Z)} (ϕ) = VΛV^{-1} + \begin{bmatrix} 1 & -θ' \\ 0 & -θ' \end{bmatrix}
\]

For the left-hand side of Eq. (26),

\[
\frac{\partial RB_{\text{pol}}/B_ϕ}{\partial (R, Z)} := \frac{1}{B^2_ϕ} \left[ \frac{∂_R (RB_R) B_ϕ - RB_R ∂_R B_ϕ}{∂_R (RB_R) B_ϕ - RB_R ∂_R B_ϕ} \right]
\]

\[
\frac{\partial X_{\text{pol}}}{\partial (R, Z)} := \frac{1}{B^2_ϕ} \left[ \frac{∂_R (RB_R) B_ϕ - RB_R ∂_R B_ϕ}{∂_R (RB_R) B_ϕ - RB_R ∂_R B_ϕ} \right]
\]
$B_\phi$ has been defined to be $R_0 B_{\phi 0}/R$ to simulate a tokamak, so we have $\partial R B_\phi = -B_\phi/R$. Therefore,

$$\frac{\partial R B_{pol}/B_\phi}{\partial (R, Z)} = \frac{1}{B_\phi} \left[ 2B_R + R \partial R B_R R \partial Z B_R \right] = 2 \frac{B_R}{B_\phi} \left[ B_R 0 \right] + \frac{R}{B_\phi} \left[ \partial R B_R \partial Z B_R \right].$$

(29)

We have preset the trajectory Eq. (23) by known $\lambda_u, \lambda_s, \theta_u(\phi), \theta_s(\phi)$, which means the right-hand side of Eq. (26) is what we can control. Based on that, we can calculate the left-hand side of Eq. (26), i.e. the field we want to construct, after which we acquire the linearized $B_R, B_Z$ field (note $B_R, B_Z$ at the axis have been preset to zero).

$$\begin{bmatrix} B_R \\ B_Z \end{bmatrix} = \begin{bmatrix} \partial R B_R & \partial Z B_R \\ \partial R B_Z & \partial Z B_Z \end{bmatrix} \begin{bmatrix} R - R_0 \\ Z - Z_0 \end{bmatrix} + \ldots
= \frac{B_{\phi 0}}{R_0} \left( VAV^{-1} + \begin{bmatrix} \theta' & -\theta \\ \theta & \theta' \end{bmatrix} \right) \begin{bmatrix} R - R_0 \\ Z - Z_0 \end{bmatrix} + \ldots. \tag{30}$$

Hence, we finish our construction. An example of Möbiusian saddle cycle is shown in Fig. 1. The non-Möbiusian case is not shown, because it is too easy to imagine. Some readers may wonder the higher order terms $\frac{\partial^k B_{pol}}{\partial (R, Z)^k}, k \geq 2$. They are determined by $\frac{\partial^k R B_{pol}/B_\phi}{\partial (R, Z)^k} = 0$ in Eq. (25).

4.2 Dummy X-cycle

In this subsection, $R_0, Z_0$ in the last model are replaced (see the orbit Eq. (23)) with $R_c(\phi), Z_c(\phi)$, two functions dependent on $\phi$, respectively. Here, an example expression of $R_c(\phi), Z_c(\phi)$ is given (the author can also design your own cycle),

$$R_c(\phi) = R_{ell} \cos (\phi + \theta_0) + R_{ax}, \tag{31}$$

$$Z_c(\phi) = Z_{ell} \sin (\phi + \theta_0) + Z_{ax}, \tag{32}$$

where the subscript $c$ is short for cycle, ell for elliptic, and ax for axis. Set $B_\phi = B_\phi(R) = R_{ax} B_{\phi ax}/R.$

The zeroth order of field expansion are denoted by $B_{Rc}, B_{Zc}, B_{\phi c}$, to highlight that they are evaluated on the cycle, e.g. $B_{\phi c} = B_\phi(R_c(\phi), Z_c(\phi), \phi)$. According to the spiral cycle equation defined above, we obtain

$$X_R = R_c B_{Rc}/B_{\phi c} = -i R_{ell} \sin (\phi + \theta_0) = R'_c(\phi) \tag{33}$$

$$X_Z = R_c B_{Zc}/B_{\phi c} = i Z_{ell} \cos (\phi + \theta_0) = Z'_c(\phi). \tag{34}$$

The first order of expansion, $\partial B_{pol}/\partial (R, Z)$, is calculated by Eq. (29).

$$\frac{\partial B_{pol}}{\partial (R, Z)} = \frac{\partial R B_R}{\partial R B_Z} \begin{bmatrix} 2B_R \\ 2B_Z \end{bmatrix} = \frac{B_\phi}{R} \left( \frac{\partial R B_{pol}/B_\phi}{\partial (R, Z)}(\phi) \right) \begin{bmatrix} 2B_R/B_\phi \\ 2B_Z/B_\phi \end{bmatrix} \tag{35}$$
Expand the poloidal field around the cycle up to first order,

\[
\begin{bmatrix}
B_R \\
B_Z
\end{bmatrix} = \begin{bmatrix}
B_{Rc} \\
B_{Zc}
\end{bmatrix} (\phi) + \begin{bmatrix}
B_{R1} \\
B_{Z1}
\end{bmatrix} (R, Z, \phi) + \cdots
\]

\[
= \begin{bmatrix}
B_{Rc} \\
B_{Zc}
\end{bmatrix} (\phi) + \frac{\partial B_{pol}}{\partial (R, Z)} (R_c(\phi), Z_c(\phi), \phi) \begin{bmatrix}
R - R_c(\phi) \\
Z - Z_c(\phi)
\end{bmatrix} + \cdots
\]

\[
= \begin{bmatrix}
B_{\phi c} R'_c / R_c \\
B_{\phi c} Z'_c / R_c
\end{bmatrix} (\phi) + \frac{B_{\phi c}}{R_c} \begin{bmatrix}
\mathbf{V} \mathbf{A} \mathbf{V}^{-1} \\
\mathbf{b}' - \mathbf{b}'
\end{bmatrix} - \begin{bmatrix}
2 B_{Rc} / B_{\phi c} & 0 \\
2 B_{Zc} / B_{\phi c} & 0
\end{bmatrix} \begin{bmatrix}
R - R_c(\phi) \\
Z - Z_c(\phi)
\end{bmatrix} + \cdots (36)
\]

Hence, we finish our construction. A dummy non-Möbiusian X-cycle with \( q = m/n = 3/1 \) is shown in Fig. 2.

**Figure 2:** (a, b, c) show the same dummy X-cycle, of which the field is constructed by Eq. (36) with parameters:

\((R_{ax}, Z_{ax}) = (1.0, 0.0), B_{\phi, ax} = 2.5, \iota = n/m = 1/3, (R_{ol}, Z_{ol}) = (0.3, 0.5), \theta_{i/s}(\phi) = \phi + \theta_0 = \phi/3 + \pi/2 \pm \pi/9, \lambda_{i/s} = e^{\pm 1/5}.\) (a) shows it from the top view, (b, c) show it from the other view. (a, b) draw arrows for the two eigenvectors of \( D\mathcal{P}^3(\phi) \) and their opposite (blue for \( \lambda < 1 \), red for \( \lambda > 1 \)), which is acquired by \( D\mathcal{P}^{\pm m} \) evolution formula and shall overlap with \( \theta_{i/s}. \) The sprouts of four invariant branches are plotted (orange for \( \mathcal{W}^u \), blue for \( \mathcal{W}^s \)). In (a, b), the manifolds are not shown on \( \phi \in [6\pi - 2/3\pi, 6\pi], \) in order not to shelter the eigenvector arrows. A transparent torus with the corresponding elliptic section is drawn for reference.
4.3 A real-world example

The equilibrium field of EAST #103950 shot at 3500ms given by EFIT is taken as background, superimposed with a non-axisymmetric field induced by the RMP (resonant magnetic perturbation) coils running in \( n = 1 \) mode. The plasma response is not considered here for simplicity. \( B_\phi \) of this shot is negative everywhere, and \( B_{pol} \) at R-Z cross-sections is clockwise.

On locating the periodic points of the Poincaré map, the simplest discrete Newton method \( x_{j+1} = x_j - h \left[ DG \left( x_j \right) \right]^{-1} G \left( x_j \right) \) is employed [40], where \( G(x) := F(x) - I \), and \( I \) is the identity map. Our map \( F \) is, of course, the Poincaré map \( P(\phi) \) at the section \( \phi \). With regard to a cycle of \( m \) toroidal turn(s), \( DP^m \) is computed by simply integrating Eq. (4) for \( 2m\pi \) radians.

After locating the cycle, one needs to know the \( DP^m(\phi) \) at every section, which is the job of \( DP^{\pm m} \) evolution formula. This formula is a traditional matrix ODE system. The Python function \texttt{scipy.integrate.solve_ivp} and the Julia package \texttt{DifferentialEquations.jl} have prepared a lot of numeric algorithms for such ODEs. Readers are free to choose the method they like. Next is to eigen decompose \( DP^m(\phi) \), which can be done by, for example, the Python function \texttt{scipy.linalg.eig}. The two eigenvectors of \( DP^m(\phi) \) and their opposite are the directions towards which the invariant manifolds grow at the beginning.

Recall that our invariant manifold growth formula,

\[
\frac{\partial X^{u/s}}{\partial s} = \left( \frac{RB_{pol}}{B_\phi} - \frac{\partial X^{u/s}}{\partial \phi} \right) \left/ \pm \| \cdots \|_2 \right. ,
\]

(20 revisited)

only requires two variables on the right-hand side, \( RB_{pol}/B_\phi \) and \( \partial X^{u/s}/\partial \phi \). We always linearly interpolate \( RB_{pol}/B_\phi \) on a regular grid of shape \([n_R, n_Z, n_\phi] \). Someone else may want to try more accurate interpolation techniques, but we have been satisfied with the linear one. For \( \partial X^{u/s}/\partial \phi \), different people have different ways to handle it numerically. In the following two subsections, two approaches to grow manifolds are exhibited.

4.3.1 Naive field line tracing

The simpler scheme is to distribute a line of Poincaré seed points along the eigenvector of a periodic point. To compute the arc length \( s \) of a manifold \( W^{u/s}(\mathcal{P}, x) \) requires the FLT trajectories used to construct the manifold be ordered, which is achieved with the assistance of the \( DP^m(\phi) \) eigenvalues in this paper.

Suppose \( x_0 \) is a saddle fixed point of the 2D Poincaré map \( \mathcal{P} \) at an R-Z section. Denote the unstable eigenvalue and eigenvector of \( DP(x_0) \) by \( \lambda_u \) and \( v_u \). Seed a line of points \((x_1, \ldots, x_N)\) along \( v_u \), equally spaced, i.e. \( x_{i+1} - x_i \) is a constant for \( 0 \leq i \leq N - 1 \). If \( \lambda_u \leq N \), \( \mathcal{P}(x_1) \) probably falls behind \( x_N \), i.e. the \( s \) of \( \mathcal{P}(x_1) \) is smaller than that of \( x_N \). This makes it difficult to compute \( s \), because the order of \( s \) is not certain. Let \( \mathcal{X} \) be a sequence defined by

\[
\mathcal{X} := (x_1, \ldots, x_N) \sim (\mathcal{P}(x_1), \ldots, \mathcal{P}(x_N)) \sim \cdots \\
= (x_1, \ldots, x_N, \mathcal{P}(x_1), \ldots, \mathcal{P}(x_N), \mathcal{P}^2(x_1), \ldots, \mathcal{P}^2(x_N), \ldots).
\]

(37)

It is expected that the \( s \) of \( \mathcal{P}(x_1) \) be greater than that of \( x_N \), the \( s \) of \( \mathcal{P}^2(x_1) \) be greater than that of \( \mathcal{P}(x_N) \), and so on, i.e. the \( s \) of \( \mathcal{P}^{k+1}(x_1) \) be greater than that of \( \mathcal{P}^k(x_N) \). In fact, as long as
the $s$ of $P(x_1)$ is greater than that of $x_N$, the conditions for $k \geq 1$ are naturally satisfied. Now that $P(x) \approx x_0 + \lambda_u(x - x_0)$ in the neighborhood of $x_0$ if $x$ is in the direction of $v_u$, one simply needs to put $x_N$ closer to $x_0$ than $P(x_1)$, so that $s(P^k(x_N)) < s(P^{k+1}(x_1))$. By virtue of this fact, the trajectories are untangled and one gets rid of tentative extending manifold of England method [11], which needs to decrease growth step when the local manifold curvature is large. In other words, it is ensured that the sequence

$$s(\mathcal{X}) = (s(x_1), \ldots, s(x_N), s(P(x_1)), \ldots, s(P(x_N)), \ldots)$$

is a strictly increasing sequence, which further guarantees that it is safe to compute $s$ by simply accumulating the lengths of segments.

The invariant manifolds in Fig. 3 and Fig. 4 are grown by naive FLT, of which the computed arc lengths $s$ are expressed by the varying color to let it be more easy-to-understand. One can immediately observe the confinement of this equilibrium relies mostly on the invariant manifolds of the lower X-cycle $\gamma_{low}$, although it also has one X-cycle at top. It is also known as the disconnected double-null configuration.

The blue arrows in Fig. 4(c) are drawn by our invariant manifold growth formula, which takes $\partial X^{u/s}/\partial \phi$ and then gives $\partial X^{u/s}/\partial s$. Evidently, $\partial X^{u/s}/\partial s$ is the growth direction of a manifold. We employ the first order central scheme to calculate $\partial X^{u/s}/\partial s$ $(s, \phi)$ by $X^{u/s}(s, \phi)$ at the two neighboring R-Z sections at $\phi \pm \epsilon$,

$$\frac{\partial X^{u/s}}{\partial \phi}(s, \phi) \approx \frac{X^{u/s}(s, \phi + \epsilon) - X^{u/s}(s, \phi - \epsilon)}{2\epsilon}.$$  

We do not encourage readers to compute a manifold to an infinite length, although one can do so by this naive method. According to Poincaré-Birkhoff and KAM theories, rational tori breed
elliptic and hyperbolic periodic points under perturbation. Near the elliptic periodic points, new
nested tori emerge. These secondary nested tori would be broken under stronger perturbation,
which forms a complicated self-similar structure. Instead, it is suggested that, for researchers
working in the fusion domain, most attention should be paid to the primary manifold structure,
\textit{i.e.} the primary islands instead of the secondary islands around the primary ones. From our
perspective, how to compute the closest surviving torus to a chaotic region is more important
than computing manifolds to infinity length.

4.3.2 Discretize the invariant manifold growth PDE to an ODE system

The other numeric way is to transform the \textit{invariant manifold growth} formula, a partial differential
equation (PDE) including $\partial_s$ and $\partial \phi$, to a system of ODEs with $s$ as the evolution parameter.
Transect the invariant manifold by $N$ \textit{R-Z cross-sections} at $(\phi_i)_{i=1}^{N}$ to discretize it. The \textit{R} and \textit{Z}
coordinates of the manifold at each section contribute two variables of the system, $X_R^{u/s}(s,\phi_i)$ and
$X_Z^{u/s}(s,\phi_i)$, totally $2N$ variables contributed by $N$ sections. With the parameter $\phi$ discretized to
a sequence $(\phi_i)_{i=1}^{N}$, $X_R^{u/s}(s,\phi_i)$ and $X_Z^{u/s}(s,\phi_i)$ become univariate functions dependent on $s$. The
manifold growth formula is thereby reduced to a $2N$-dim ODE system.

$$\frac{dX_R^{u/s}(s,\phi_1)}{ds} = \left( \frac{RB_{pol}}{B_\phi} - \frac{\Delta X_R^{u/s}(s,\phi_1)}{\Delta \phi} \right) / \pm \cdots \|_2,$$

$$\frac{dX_R^{u/s}(s,\phi_2)}{ds} = \left( \frac{RB_{pol}}{B_\phi} - \frac{\Delta X_R^{u/s}(s,\phi_2)}{\Delta \phi} \right) / \pm \cdots \|_2 \quad \text{(20 discretized)}$$

$$\cdots$$

$$\frac{dX_R^{u/s}(s,\phi_N)}{ds} = \left( \frac{RB_{pol}}{B_\phi} - \frac{\Delta X_R^{u/s}(s,\phi_N)}{\Delta \phi} \right) / \pm \cdots \|_2,$$

where $\Delta X_R^{u/s}(s,\phi_i) / \Delta \phi$ denotes a numeric alternative of $\partial X_R^{u/s}(s,\phi) / \partial \phi$. For example, it can be
defined as the first- or second-order central difference schemes,

$$\frac{\Delta X_R^{u/s}(s,\phi_i)}{\Delta \phi} := \frac{X_R^{u/s}(s,\phi_{i+1}) - X_R^{u/s}(s,\phi_{i-1})}{2\Delta \phi},$$  \hfill (40)

$$\frac{\Delta X_R^{u/s}(s,\phi_i)}{\Delta \phi} := \frac{X_R^{u/s}(s,\phi_{i+1}) - 2X_R^{u/s}(s,\phi_i) + X_R^{u/s}(s,\phi_{i-1})}{\Delta \phi^2},$$  \hfill (41)

if $(\phi_i)_{i=1}^{N}$ is an arithmetic sequence with the common difference $\Delta \phi = \phi_{i+1} - \phi_i$.

This discretizing method works fine for the invariant manifolds of outmost saddle cycles, but
not so for the $q = m/n = 3/1$ island chain. The primary reason is that the two eigenvectors are
too close to each other for \textit{X}-cycles of island chains. Although this scheme has such a drawback,
the authors still think it is worth introducing, because this scheme directly utilizes the \textit{invariant
manifold growth} formula.

4.3.3 Denotations and notions explained with the aid of figures
Our denotations are carefully selected to enhance one’s expressing ability on the relevant subjects. For example, a periodic orbit for a map is best seen as a whole. Then, we can use \( \mathcal{W}^u(P, \{x_1, x_2, x_3\}) \) to denote all the unstable branches of the X-cycle of the \( g = m/n = 3/1 \) island chain in Fig. 4. This X-cycle has three striking points \( \{x_1, x_2, x_3\} \) through \( \phi = 0 \) section, each of which has two stable and two unstable branches. Totally there are 6 branches for \( \mathcal{W}^s(P, \{x_1, x_2, x_3\}) \) and 6 branches for \( \mathcal{W}^u(P, \{x_1, x_2, x_3\}) \). One can also specify in a more fine-grained way by replacing \( P \) with \( P^m, \{x_1, x_2, x_3\} \) with \( x_i \), that is \( \mathcal{W}^{u/s}(P^3, x_i) \), which represents the two unstable/stable branches belonging to \( x_i \), respectively. Obviously,

\[
\mathcal{W}^{u/s}(P, \{x_1, x_2, x_3\}) = \bigcup_{i \in \{1,2,3\}} \mathcal{W}^{u/s}(P^3, x_i).
\]

(42)

Recall that it was mentioned in Sect. 2.2, the 2D manifold \( \mathcal{W}^{u/s}(B, \gamma) \) consists of all the corresponding 1D manifolds \( \mathcal{W}^{u/s}(P^m(\phi), x(\phi)) \), i.e.

\[
\mathcal{W}^{u/s}(B, \gamma) = \bigcup_{\phi \in [0,6\pi]} \{ x(R, Z, \phi) | (R, Z) \in \mathcal{W}^{u/s}(P^3(\phi), x(\phi)) \},
\]

(9 revisited)

where \( x(\phi) \) is a 6\( \pi \)-periodic function representing the \( (R, Z) \) coordinates of this X-cycle \( \gamma \). \( P(x_0) = x_1 \), \( P(x_1) = x_2 \), \( P(x_2) = x_3 \). Then, \( x(0) = x_1 \), \( x(-2\pi) = x_2 \), \( x(-4\pi) = x_3 \). \( B_{\theta} \) is negative in this shot.) Viewing the 3D Fig. 3 and 2D Fig. 4 together can help understand the relationship between 2D manifolds \( \mathcal{W}^{u/s}(B, \gamma) \) and 1D manifolds \( \mathcal{W}^{u/s}(P^m(\phi), x(\phi)) \).

Some regions enclosed by the invariant manifolds of \( \gamma_{\text{low}} \) are filled by colored scatter points. Points are marked with distinctive colors, and then mapped by \( P^{-1} \). How these regions are connected by field lines is reflected through the color pattern of scatter points. All the points of transversal intersection of the two manifolds \( \mathcal{W}^u(P, x_{\text{low}}) \) and \( \mathcal{W}^s(P, x_{\text{low}}) \) (shown by the red and green curves, respectively) are homoclinic to \( x_{\text{low}} \). There are many homoclinic orbits when a homoclinic intersection happens, two of which are the most important. As shown in Fig. 4(a, b), the two endpoints of colored regions are surrounded by yellow and dark blue scatter points, respectively, which represent the two principal homoclinic orbits.

Although the fluxes in the colored regions filled by scatter points are the same, e.g. the \( 1 \) and \( 2 \) regions in Fig. 4(b), this flux value probably differs from that of the uncolored regions, e.g. \( 3 \) and \( 4 \). One should always be careful of this fact.
Invariant Manifold Growth

Figure 4: (a) Poincaré plot at $\phi = 0$ of EAST #103950 shot at 3500 ms (EFIT + vacuum RMP). Some invariant manifolds are grown and plotted. (b) Enlarged view of (a) near the lower X-point. Dense scatter points with color are presented to show how the regions are connected by field lines. (c) Enlarged view of (b) near $x_1$. The blue arrows are drawn according to Eq. (39).
5. Comparisons with existing works

To save readers’ time on comparing various approaches developed to study invariant manifolds, this section delivers how our work is different from other existing ones, albeit obvious. The comparisons of our \( \mathcal{DP}^m \) evolution and invariant manifold growth formulas with others’ works are presented in Sect. 5.1 and 5.2, respectively.

5.1 Works similar to our \( \mathcal{DP}^m \) evolution formula

As far as the authors know, most of existing research has been satisfied with the Floquet’s normal form. Floquet theory is so successful that people cease further exploration. Floquet theorem is paraphrased here, which can be found in any ODE textbook:

Let \( \dot{x} = A(t)x \) be a real linear system of ordinary differential equations, where \( A(t) \) is \( T \)-periodic and piecewise continuous. Let \( F(t, t_0) \) be its fundamental solution. Then, for each complex matrix \( B \) such that \( e^{TB} = F(T, 0) \), there is a complex \( T \)-periodic matrix function \( P(t) \) such that

\[
F(t, 0) = P(t)e^{tB} \text{ for all } t \in \mathbb{R},
\]

which is called a Floquet normal form for the fundamental solution \( F \).

An important corollary of Floquet theorem is that the system can be reduced to one with constant coefficients, as repeated below:

The system above \( \dot{x} = A(t)x \) can be reduced by a linear coordinate transform \( x = P(t)y \) to a linear \( T \)-periodic system of differential equations:

\[
\dot{y} = By.
\]

The combination of the theorem and corollary above is sometimes called Floquet-Lyapunov theorem, which governs how \( F(t, 0) \) varies with \( t \) and guides how to solve the system in a simpler way. However, it remains unclear how \( F(t_0 + T, t_0) \) changes with \( t_0 \), which is essential to lessen the computational resources needed to solve for the initial growth directions of stable and unstable manifolds. Recall that \( F(t_0 + T, t_0) \) is indeed the same thing as our \( \mathcal{DP}^m(\phi) \). To the best of our knowledge, the most similar work to our \( \mathcal{DP}^m \) evolution formula is given by Tsutumi [41]. Tsutumi considered the following linear system:

\[
\frac{dx}{d\phi} = A(\phi, t)x, \quad -\infty < \phi, t < +\infty, \quad (1.1) \text{ in [41]}
\]

where \( x = x(\phi, t) \) is a complex \( n \)-column vector and \( A(\phi, t) \) is a complex \( n \times n \) matrix function \( T \)-periodic in \( \phi \). Theorem 1. in [41] is repeated as below:

There exists a monodromy matrix of (1.1) which does not depend on \( t \) \(^7\) if and only if there exists a matrix function \( \Gamma(\phi, t) \) which is defined on \( \infty < \phi, t < \infty \), \( T \)-periodic in \( \phi \), and satisfies

\[
\frac{\partial}{\partial t} A(\phi, t) - \frac{\partial}{\partial \phi} \Gamma(\phi, t) + [A(\phi, t), \Gamma(\phi, t)] = 0 \quad (1.2) \text{ in [41]}
\]

\(^7\)In [41], “there exists a monodromy matrix of (1.1) which does not depend on \( t \)” is equivalent to that “the internal structure of every monodromy matrix of (1.1) does not depend on \( t \).” By internal structure, Tsutumi means the characteristic multipliers of a matrix and their algebraic and geometric multiplicities.
Eq. (1.2) is more complicated than our \( DP^{\pm m} \) *evolution* formula (16) because \( A(\phi, t) \) relies on both \( \phi \) and \( t \). If \( \partial A / \partial t \) vanishes, one can easily observe that \( \Gamma \) is governed by the same equation as \( DP^{\pm m} \) *evolution* formula. Tsutsumi did not explain in [41] what \( \Gamma \) can be and how to construct it. His attention was paid to the condition of \( A(\phi, t) \) such that the monodromy matrix does not depend on \( t \). This condition is revealed in his theorem by the existence of the matrix function \( \Gamma \) which satisfies Eq. (1.2).

5.2 Works similar to our invariant manifold growth formula

The works similar to our invariant manifold formula are collated into Table 1, of which the work of S. S. Abdullaev is discussed in detail (in this subsection and Appendix B). The classical invariance equation (1.7) in [9] for \( n \)-dim maps might be too general for fusion scientists to be useful. T. E. Evans, J. P. England, and J. M. Ottino *et al.* focused on numerical or experimental methods rather than analysis.

Abdullaev [29] analyzes a magnetic field which is not so axisymmetric that it can be decomposed into two parts, \( B(R, Z, \phi) = B_0(R, Z) + B_{\text{perb}}(R, Z, \phi) \), where \( B_0 \) is axisymmetric. Under this condition, it is easy to transform the variables of the FLT ODE system under \( B_0 \) from \( (R, Z) \) to the canonically conjugated variables \( (z, p_z) \). However, this condition limits the range of application of Abdullaev’s equations to the tokamak cases where the 3D perturbation is small enough. Then, after the FLT ODE system is transformed to a Hamiltonian form (the normalized poloidal flux \( \psi_p \) as the Hamiltonian), the impact of the non-axisymmetric part is considered by a perturbation \( \epsilon\psi_p^{(1)}(z, p_z, \phi) \) to the Hamiltonian, which implies a numeric step to transform \( B_{\text{perb}}(R, Z, \phi) \) to \( \epsilon\psi_p^{(1)}(z, p_z, \phi) \). In contrast, our theory always stays in the standard cylindrical coordinate system.

Then, Abdullaev acquires an expression of Poincaré map \( (\phi_k, \psi_k) \mapsto (\phi_{k+1}, \psi_{l+1}) \) for a full *poloidal* turn through the Poincaré integral, which is an integral of \( \psi_p^{(1)} \) along the unperturbed trajectory for a poloidal turn. Although it is a classical approach to express the Poincaré map, the equality of the relevant equations only applies to the cases where \( \epsilon \) is infinitesimal, because this approach is essentially a first order approximation to the Poincaré map.

Based on the expressions for Poincaré map, an *implicit* expression \( F^{(s,u)}(\phi, \psi) = 0 \) of the subsets of stable and unstable manifolds intersected by the Poincaré section \( \Sigma_0 \) is acquired by letting \( \psi_{k+1} = 0 \) at \( \phi_{k+1} \to \pm \infty \), respectively. In our theory, an explicit parameterization is exhibited by the *invariant manifold growth* formula. Furthermore, for those points on stable and unstable manifolds but not on \( \Sigma_0 \), one needs to map them back to \( \Sigma_0 \), which is not needed in our theory. Note that one variable of \( F^{(s,u)}(\phi, \psi) \) is \( \psi \), which has been perturbed by \( \epsilon\psi_p^{(1)} \). Transforming \( \psi \) back to spatial coordinates brings additional error. Lastly, Eq. (70) in [29] (shown in Table 1) is deduced as the final implicit form for stable and unstable manifolds. However, we remark that the \((x, y)\) coordinates appearing in the implicit form \( F^{(s,u)}(\phi, x, y) = 0 \) are not the standard \((R, Z)\) cylindrical coordinates. The \( x \) - and \( y \) -axes are even probably not perpendicular. A more detailed comparison equipped with equations is put in Appendix B.
<table>
<thead>
<tr>
<th>Method</th>
<th>Manifold and dynamical system dimensions</th>
<th>Typical manifold expression</th>
</tr>
</thead>
</table>
| Ours                       | 2D manifolds of 3D flows                 | \[
\frac{\partial \mathbf{X}^{\omega}}{\partial \mathbf{s}} = \frac{\mathbf{R}B_{\mathbf{s}}}{\mathbf{B}} - \frac{\partial \mathbf{X}^{\omega}}{\partial \mathbf{\phi}} \pm \frac{\mathbf{RB}_{\mathbf{\phi}}}{\mathbf{B}} \frac{\partial \mathbf{X}^{\omega}}{\partial \mathbf{\phi}} \] (20 revisited) |
| S.S. Abdullaev et al.      | 2D manifolds of 3D flows [29]            | \[ F^{(n)}(\phi, x, y) = \gamma xy \] 
\[ \pm \epsilon \frac{\partial}{\partial \phi} p \left( \phi \pm \frac{1}{2} \ln \frac{Q}{\gamma (xy)} - \frac{1}{2} \ln \frac{\gamma (xy)}{y} \right) = 0 \] (70) in [29] |
| Classical invariance Eq.   | \( n \)-D maps [9]                      | \( F(W(s)) - W(f(s)) = 0 \) (17) in [9] |
| T.E. Evans et al.          | 2D manifolds of 3D flows [7, 14, 28]     | Pure numeric. See Sect 3 of [28]. "Invariant manifolds associated with a hyperbolic fixed point of the \( \text{T}_{3} \) map, such as \( \times_{i} \) in figure 2, are calculated by constructing the (un)stable eigenvectors \( \mathbf{\epsilon}_{(a), l} \) associated with a fixed point \( \times \) and populating a line segment directed along \( \mathbf{\epsilon}_{(a), l} \) with a set of evenly distributed points." |
| J.P. England et al.        | 1D manifolds of 2D maps [11], 1D manifold of \((n-1)\)-D Poincaré map induced by \( n \)-D flow [12], 2D manifolds of 3D flows [13] | Pure numeric |
| J.M. Ottino et al.         | 2D manifolds of 3D maps [26]             | Ottino et al. visualized the invariant manifolds schematically by means of a fluid mechanical experiment. In [36], the caption of Fig. 5.6.3 writes "Visualization of manifolds \( W'(P) \) and \( W''(P) \) by means of a flow visualization (thought) experiment . . . ."; the caption of Fig. 5.8.3 writes "Visualization of manifolds \( W(P) \) in a time-periodic system by means of a fluid mechanical experiment. . . ."; a sentence around Fig. 5.8.3 writes "Figure 5.8.3 (which is a continuation of Figure 5.6.3) shows, schematically, intermediate times . . ." |
6. Discussion and conclusion

Although the magnetic fields in tokamaks are mostly considered axisymmetric, almost all kinds of auxiliary heating schemes, except the ohmic heating of the central solenoid, are strongly localized and non-axisymmetric. The topology of magnetic field plays a dominant role in the behaviour of the confined plasma and the scrape-off layer. Motivated by the curiosity about the intrinsic characteristics of general 3D vector fields (not necessarily divergence-free), an analytic theory on the invariant manifolds of cycles is established in this paper, where the short-period cycles are regarded the skeleton of fields.

The invariant manifolds of saddle and parabolic cycles grow from the eigenvectors of $D\mathcal{P}_m(\phi)$. By $D\mathcal{P}_m$ evolution formula (16), one gets rid of repetitive integration of Eq. (4) for $D\mathcal{P}_m$ at every R-Z section. The primitive FLT ODE system (3) is extended to the invariant manifold growth formula in cylindrical coordinates (20) by analyzing the relevant differentials.

The stable and unstable manifolds interweave for infinite times into a tangle when a transverse intersection happens. The regions involved are intrinsically chaotic, for which it is suggested to call them chaotic field regions. The name ergodic field is not encouraged due to the well-known fact that in these regions periodic orbits are dense, which are not ergodic. Heteroclinic intersection is more complicated than homoclinic intersection. For fusion scientists, heteroclinic intersection means that the neighboring island chains partially or totally overlap and trample each other, which usually induces a larger mixing region than pure homoclinic intersection.

It is proposed to use the notion of invariant manifolds of outmost saddle cycle(s) to characterize the magnetic topology at the plasma edge when the field is strongly non-axisymmetric. If a tokamak operates in the single-null (resp. double-null, or somehow more strange) mode, there

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8To be accurate, we refer readers to Pugh’s general density theorem: “Generically, the set of periodic points of a $C^1$ diffeomorphism is dense in its nonwandering set.” [42]
is one (resp. two, or more) outmost saddle cycle(s). The transversely intersecting manifolds are suspected of causing the spiral ribbon-like pattern of heat deposition on the divertor found in tokamak experiments, which needs further verification. In the scrape-off layer, how to disperse the particle and heat fluxes before they land on the divertor is also an interesting problem, worthy of more investigation. Note that various single particle drift effects are not considered in this paper. The $E \times B$ drift is easy to be taken into account since $v_E = E \times B/B^2$ is independent of the particle velocity. One only needs to substitute $B$ in our formulas for $B + v_E$. In contrast, the $\nabla B$ and curvature drift velocities depend on $v_\perp$ and $v_\parallel$, respectively, which await more research. With the drifts taken into account, the heat load pattern observed by diagnostics like infrared thermography is supposed to be more consistent with the divertor regions covered by the invariant manifolds of outmost saddle cycles. For both tokamak and stellarator communities, the transport issue at the plasma boundary is essential to control the heat load below the material limit.

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**APPENDIX A. PROOFS**

A.1 The geometric meaning of $|DX_{pol}(\phi_s, \phi_e)|$

$$|DX_{pol}(\phi_s, \phi_e)| = \exp \left( \int_{\phi_s}^{\phi_e} \frac{R(\nabla \cdot B)}{B_\phi} d\phi \right) \frac{B_\phi|_{\phi_e}}{B_\phi|_{\phi_s}} \quad (12 \text{ revisited})$$

Proof.

$$|DX_{pol}| = e^{\int_{\phi_s}^{\phi_e} v_{Adf} d\phi} = \exp \left( \int_{\phi_s}^{\phi_e} \partial_R (RB_R/B_\phi) + \partial_Z (RB_Z/B_\phi) d\phi \right)$$

$$= \exp \left( \int_{\phi_s}^{\phi_e} \frac{1}{B_\phi} \left( RB_R \partial_R B_\phi + \partial_Z B_Z \right) + B_R B_\phi - R(B_R \partial_R + B_Z \partial_Z) B_\phi \right)$$

$\nabla \cdot B = \partial_R B_R + \partial_Z B_Z + 1/R \left( B_R + \partial_\phi B_\phi \right)$

$$= \exp \left( \int_{\phi_s}^{\phi_e} \frac{1}{B_\phi} \left( RB_R (\nabla \cdot B) - B_\phi \partial_R B_\phi - R(B_R \partial_R + B_Z \partial_Z) B_\phi \right) d\phi \right) \quad (48)$$

To further simplify the expression above, we need to prepare the following differentials.

$$\nabla B_\phi = \hat{e}_R \partial_R B_\phi + \hat{e}_Z \partial_Z B_\phi + \hat{e}_\phi \frac{1}{R} \partial_\phi B_\phi$$

$$d\phi = \hat{e}_R dR + \hat{e}_Z dZ + \hat{e}_\phi dB_\phi$$

$$\nabla B_\phi \cdot d\phi = \partial_R B_\phi \hat{e}_R dR + \partial_Z B_\phi \hat{e}_Z dZ + \partial_\phi B_\phi d\phi$$

Now we can reorganize the terms at the right-hand side of Eq. (48) in a much more compact way.
\begin{align}
|DX_{\text{pol}}| &= \exp \left( \int_{\phi}^{\phi_*} \left( \frac{R (\nabla \cdot B)}{B} - \frac{1}{B} \left( \partial_{\phi} B_{\phi} + \frac{R B_{\phi}}{B} \partial_{\phi} \frac{B_{\phi}}{B} + \frac{R B_{z}}{B} \partial_{\phi} \frac{B_{z}}{B} \right) \right) d\phi \right) \\
&= \exp \left( \int_{\phi}^{\phi_*} \left( \frac{R (\nabla \cdot B)}{B} - \nabla B_{\phi} \cdot dI \right) \right) \\
&= \exp \left( \int_{\phi}^{\phi_*} \left( \frac{R (\nabla \cdot B)}{B} - \nabla B_{\phi} \cdot dI \right) \right) = \exp \left( \int_{\phi}^{\phi_*} \left( \frac{R (\nabla \cdot B)}{B} - \ln B_{\phi} \right) \right) \tag{53} \\
&= \exp \left( \int_{\phi}^{\phi_*} \left( \frac{R (\nabla \cdot B)}{B} \right) B_{\phi} \bigg|_{\phi} \right) \tag{54}
\end{align}

(Here \( \ln \) is the complex logarithm instead of the real one. So it can handle negative numbers.)

\begin{align}
&= \exp \left( \int_{\phi}^{\phi_*} \left( \frac{R (\nabla \cdot B)}{B} \right) B_{\phi} \bigg|_{\phi} \right) \tag{55}

\end{align}

\begin{align}
\text{A.2} \quad \mathcal{D}X \text{ to } \mathcal{D}P \text{ formula}

& \quad \begin{bmatrix}
1 & -\frac{R B_{\phi}}{B}\bigg|_{\text{end}} \\
1 & -\frac{R B_{z}}{B}\bigg|_{\text{end}}
\end{bmatrix} C_{\phi}^{-1} DX = \begin{bmatrix} DX_{\text{pol}} \end{bmatrix} \cdot \begin{bmatrix} * \\
* 
\end{bmatrix}. \tag{14 \text{ revisited}}
\end{align}

\( C_{\phi} \) and \( C_{\phi} \) are the matrix \[
\begin{bmatrix}
\cos \phi & -R \sin \phi \\
\sin \phi & R \cos \phi \\
1 & 0
\end{bmatrix}
\]
evaluated at the starting and ending points, respectively.

\text{Proof.} Consider the differential relationship in cylindrical coordinates,

\begin{align}
\begin{bmatrix}
dx \\
dy \\
dz
\end{bmatrix} &= \begin{bmatrix}
\frac{dx}{d\phi} & R \cos \phi \\
\frac{dy}{d\phi} & R \sin \phi \\
\frac{dz}{d\phi}
\end{bmatrix} \\
\begin{bmatrix}
dx \bigg|_{\text{start}} \\
\frac{dx}{d\phi} \\
\frac{dz}{d\phi}
\end{bmatrix} &= \begin{bmatrix}
R \cos \phi & R \sin \phi \\
R \sin \phi & R \cos \phi \\
1 & 0
\end{bmatrix} \\
\begin{bmatrix}
dx \bigg|_{\text{end}} \\
\frac{dx}{d\phi} \\
\frac{dz}{d\phi}
\end{bmatrix} &= \begin{bmatrix}
\cos \phi & -R \sin \phi \\
\sin \phi & R \cos \phi \\
1 & 0
\end{bmatrix} \begin{bmatrix}
dx \bigg|_{\text{end}} \\
\frac{dx}{d\phi} \\
\frac{dz}{d\phi}
\end{bmatrix}
\end{align}

Suppose the solution \( X(x_0, t) \) crosses the R-Z cross-section \( \phi_e \) at the time \( T \). We write the relevant differential equation \( dX(x_0, T) = DX(x_0, T) \cdot dx_0 \) as

\begin{align}
\begin{bmatrix}
dx \\
\frac{dx}{d\phi} \\
\frac{dz}{d\phi}
\end{bmatrix} = DX \\
\begin{bmatrix}
\frac{dx_0}{dx} \\
\frac{dx_0}{d\phi} \\
\frac{dx_0}{dz}
\end{bmatrix}
\end{align}

\begin{align}
\begin{bmatrix}
\cos \phi & -R \sin \phi \\
\sin \phi & R \cos \phi \\
1 & 0
\end{bmatrix} \begin{bmatrix}
\frac{dx}{d\phi} \\
\frac{dz}{d\phi}
\end{bmatrix} &= \begin{bmatrix}
\cos \phi & -R \sin \phi \\
\sin \phi & R \cos \phi \\
1 & 0
\end{bmatrix} \begin{bmatrix}
\frac{dx_0}{dx} \\
\frac{dx_0}{d\phi} \\
\frac{dx_0}{dz}
\end{bmatrix}
\end{align}

where \( x_0 \) and \( X(x_0, T) \) denote the starting and ending point, respectively. Now we vary \( x_0 \). \( X_{\text{pol}}(x_0, \phi_e, \phi_f) \) no longer necessarily corresponds to \( X(x_0, T) \), because \( X(x_0, T) \) perhaps differs by a non-zero \( dx_0 \). A similar differential analysis is
done for $X_{pol}(x_0, \phi_1, \phi_2)$ as below,

$$
\begin{align*}
\mathcal{D}X_{pol} \left[ \frac{dx_R}{dx_Z} \right]_{\text{Eq. } (59)} &= \left[ \begin{array}{c}
dX_R - \frac{RB_S}{RB_P} dX_P \\
dX_Z - \frac{RB_S}{RB_P} dX_P \\
\end{array} \right]_{\text{end}} \\
= \left[ \begin{array}{c}
1 - \frac{RB_S}{RB_P} \\
1 - \frac{RB_S}{RB_P} \\
\end{array} \right]_{\text{end}} & \left[ \begin{array}{c}
dX_R \\
dX_Z \\
\end{array} \right] \\
\mathcal{D}X_{pol} \left[ \frac{dx_R}{dx_Z} \right]_{\text{Eq. } (58)} &= \left[ \begin{array}{c}
1 - \frac{RB_S}{RB_P} \\
1 - \frac{RB_S}{RB_P} \\
\end{array} \right]_{\text{end}} C^{-1} D X C_i \\
\end{align*}
$$

(59)

(60)

(61)

The equation above always holds for any $dx_R$, $dx_Z$, and $dx_P$, hence the corresponding coefficients shall equal,

$$
\left[ \begin{array}{cc}
\mathcal{D}X_{pol} & \end{array} \right]_{\text{end}} = \left[ \begin{array}{c}
1 - \frac{RB_S}{RB_P} \\
1 - \frac{RB_S}{RB_P} \\
\end{array} \right]_{\text{end}} C^{-1} D X C_i. \\
$$

(14 revisited)

\[\square\]

### A.3 $\mathcal{D}P^{\pm m}$ evolution formula

$$
\frac{d}{d\phi} \mathcal{D}P^{\pm m}(\phi) = \left[ A(\phi), \mathcal{D}P^{\pm m}(\phi) \right]. \\
$$

(16 revisited)

**Proof.**

Integrating Eq. (4) w.r.t. $\phi$ from $\phi_i$ to $\phi_i + \Delta\phi$,

$$
\mathcal{D}X_{pol}(\phi_i, \phi_i + \Delta\phi) - \mathcal{D}X_{pol}(\phi_i, \phi_i) = \int_{\phi_i}^{\phi_i + \Delta\phi} \frac{\partial}{\partial\phi_i} \mathcal{D}X_{pol}(\phi_i, \phi_i) \ d\phi_i. \\
$$

(62)

Differentiating both sides of Eq. (62) w.r.t. $\phi_i$,

$$
\frac{d}{d\phi_i} \mathcal{D}X_{pol}(\phi_i, \phi_i + \Delta\phi) = A(\phi_i + \Delta\phi) \mathcal{D}X_{pol}(\phi_i, \phi_i + \Delta\phi) - A(\phi_i) \\
+ \int_{\phi_i}^{\phi_i + \Delta\phi} \frac{\partial}{\partial\phi_i} A(\phi_i) \mathcal{D}X_{pol}(\phi_i, \phi_i) \ d\phi_i. \\
$$

(63)

(64)

Though we know how to calculate $\frac{\partial}{\partial\phi_i} \mathcal{D}X_{pol}(\phi_i, \phi_i)$ as shown in Eq. (4), little is known about $\frac{\partial}{\partial\phi_i} \mathcal{D}X_{pol}(\phi_i, \phi_i)$. Notice that $\mathcal{D}X_{pol}(\phi_i, \phi_i)$ is the inverse matrix of $\mathcal{D}X_{pol}(\phi_i, \phi_i)$. The following Eq. (67) from [43] is useful to solve for $(K^{-1})'$ by $K(x)$ and its derivative $K'(x)$, where $K(x)$ is a univariate matrix function.

$$(KK^{-1})' = 0$$

(65)

$$= K'K^{-1} + K \left(K^{-1}\right)'$$

(66)

$$\Rightarrow \left(K^{-1}\right)' = -K^{-1}K'K^{-1}$$

(67)

We borrow the Eq. (67) to solve for $\frac{\partial}{\partial\phi_i} \mathcal{D}X_{pol}(\phi_i, \phi_i)$, as shown below

$$
\frac{\partial}{\partial\phi_i} \mathcal{D}X_{pol}(\phi_i, \phi_i) \left( DX_{pol}^{-1}(\phi_i, \phi_i) \right) = -\left( DX_{pol}^{-1}(\phi_i, \phi_i) \right) \frac{\partial}{\partial\phi_i} \mathcal{D}X_{pol}(\phi_i, \phi_i) \left( DX_{pol}^{-1}(\phi_i, \phi_i) \right) \\
= - \left( DX_{pol}^{-1}(\phi_i, \phi_i) \right) A(\phi_i) DX_{pol}(\phi_i, \phi_i) \left( DX_{pol}^{-1}(\phi_i, \phi_i) \right) \left( DX_{pol}^{-1}(\phi_i, \phi_i) \right) \\
= - DX_{pol}(\phi_i, \phi_i) A(\phi_i). \\
$$

(68)
Next, we calculate the total derivative of $DX_{\text{pol}}(\phi, \phi + \Delta \phi)$ w.r.t. $\phi$. Let $\phi_c = \phi + \Delta \phi$.

$$\frac{d}{d\phi} DX_{\text{pol}}(\phi_c, \phi, \phi + \Delta \phi) = A(\phi_c) DX_{\text{pol}}(\phi, \phi, \phi + \Delta \phi) - A(\phi) + \int_{\phi}^{\phi_c} \frac{A(\phi_c)}{d\phi} \left( A(\phi) DX_{\text{pol}}(\phi, \phi, \phi + \Delta \phi) \right) d\phi_c$$

$$= A(\phi_c) DX_{\text{pol}}(\phi, \phi, \phi + \Delta \phi) - A(\phi) - \left( DX_{\text{pol}}(\phi, \phi, \phi + \Delta \phi) - I \right) A(\phi)$$

$$= A(\phi_c) DX_{\text{pol}}(\phi, \phi, \phi + \Delta \phi) - DX_{\text{pol}}(\phi, \phi, \phi + \Delta \phi) A(\phi)$$

(69)

For orbits of $m$ toroidal turn(s), $A(\phi_c)$ and $A(\phi_c \pm 2m\pi)$ are identical since $A$ is a periodic function on the cycle.

$$A(\phi_c) = A(\phi_c \pm 2m\pi)$$

(70)

Let $\Delta \phi$ be $\pm 2m\pi$, then $DX_{\text{pol}}(\phi, \phi + \Delta \phi)$ can be substituted for $D^P_{\pm m}(\phi)$. Eq. (69) is simplified to

$$\frac{d}{d\phi} D^P_{\pm m}(\phi) = A(\phi) D^P_{\pm m} - D^P_{\pm m} A(\phi)$$

(16 revised)

One might suspect whether Eq. (16) works for $D^P_{\pm m}$, the inverse of $D^P_{\pm m}$. Consider $D^P_{\pm m} = (D^P_{\pm m})^{-1}$. By Eq. (67),

$$\frac{d}{d\phi} D^P_{\pm m}(\phi) = -(D^P_{\pm m})^{-1} (D^P_{\pm m})' (D^P_{\pm m})^{-1} = -(D^P_{\pm m})^{-1} (A D^P_{\pm m} - D^P_{\pm m} A)(D^P_{\pm m})^{-1}$$

$$= [A(\phi), D^P_{\pm m}(\phi)]$$

(71)

which indicates that the formula (16) applies for both $P^m$ and its inverse $P_{\pm m}$.

□

A.4 $D^P_{\pm m}$ eigenvector evolution formula

$$\Theta' = \left( \begin{bmatrix} 1 & -1 \\ V \Lambda - D^P_{\pm m} [1 & -1] \right)^{-1} (D^P_{\pm m})' V$$

(17 revised)

Proof. Firstly, the eigenvectors are parameterized by $\theta$ as shown below,

$$D^P_{\pm m} v_i(\phi) = \lambda_i v_i(\phi)$$

let $v_i := \begin{bmatrix} \cos \theta_i \\ \sin \theta_i \end{bmatrix}(\phi)$, $i \in \{1, 2\}$

(72)

which is safe because by the $\lambda$-invariance corollary we know the eigenvectors are not complex on saddle cycles.

Concatenate the two eigenvectors into a matrix $V$,

$$D^P_{\pm m} \begin{bmatrix} v_1(\phi), v_2(\phi) \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1(\phi), \lambda_2 v_2(\phi) \end{bmatrix} = V \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

(73)

Differentiating Eq. (73) on $\phi$,

$$\left( D^P_{\pm m} \right)' V + D^P_{\pm m} V' = V' \Lambda + V \Lambda'$$

(74)

$V'$ can be simplified by

$$V' = \begin{bmatrix} 1 & -1 \end{bmatrix} V \begin{bmatrix} \theta_1' \\ \theta_2' \end{bmatrix}$$

(75)

$\Theta$ and $\Lambda$ are diagonal and hence commute, therefore Eq. (74) becomes

$$\left( D^P_{\pm m} \right)' V = V' \Lambda - D^P_{\pm m} \cdot V' + V \Lambda'$$

(76)

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} V \Lambda \Theta' - D^P_{\pm m} \begin{bmatrix} 1 & -1 \end{bmatrix} V \Theta' + V \Lambda'.$$

(77)

For cycles, $(D^P_{\pm m})' = [A, D^P_{\pm m}]$ implies that $\lambda_1' = 0, \lambda_2' = 0$ as shown by the Eq. (19), then Eq. (77) is simplified to be

$$\left( D^P_{\pm m} \right)' V = \left( \begin{bmatrix} 1 & -1 \end{bmatrix} V \Lambda - D^P_{\pm m} \begin{bmatrix} 1 & -1 \end{bmatrix} \right) \Theta'$$

(78)

□
### A.5 Invariant manifold growth formula

\[
\frac{\partial X^{u/s}}{\partial s} = \frac{RB_{pol}}{B_{\phi}} \frac{\partial X^{u/s}}{\partial \phi} \pm \left| \frac{RB_{pol}}{B_{\phi}} \frac{\partial X^{u/s}}{\partial \phi} \right|_2
\]

(20 revisited)

**Proof.** The relationship between the differentials \(ds\), \(d\phi\), \(dR\), and \(dZ\) we will use is shown in Fig. 6. To solve for \(ds/d\phi\), the differential \(ds\) is given as below,

\[
ds^2 = \left( X_{R}^{u/s}(s, \phi) + \frac{RB_{K}}{B_{\phi}} d\phi - X_{R}^{u/s}(s, \phi + d\phi) \right)^2 + \left( X_{Z}^{u/s}(s, \phi) + \frac{RB_{Z}}{B_{\phi}} d\phi - X_{Z}^{u/s}(s, \phi + d\phi) \right)^2
\]

Dividing both sides by \(d\phi^2\),

\[
\frac{d^2 s}{d\phi^2} = \left( \frac{X_{R}^{u/s}(s, \phi) - X_{R}^{u/s}(s, \phi + d\phi)}{d\phi} \right)^2 + \left( \frac{X_{Z}^{u/s}(s, \phi) - X_{Z}^{u/s}(s, \phi + d\phi)}{d\phi} \right)^2
\]

(79)

\[
\frac{ds}{d\phi} = \pm \sqrt{\left( \frac{RB_{K}}{B_{\phi}} - \frac{\partial X_{R}^{u/s}}{\partial \phi} \right)^2 + \left( \frac{RB_{Z}}{B_{\phi}} - \frac{\partial X_{Z}^{u/s}}{\partial \phi} \right)^2}.
\]

(80)

\[
\text{(the square removed)}
\]

(81)

\[
\frac{ds}{d\phi} = \pm \left| \frac{RB_{pol}}{B_{\phi}} \frac{\partial X^{u/s}}{\partial \phi} \right|_2.
\]

(82)

---

**Figure 6:** Geometric diagram to show the relationship between the differentials \(ds\), \(d\phi\), \(dR\), and \(dZ\). It is supposed that there exists a hyperbolic cycle at bottom, from which an invariant manifolds grows. (a) shows the differential involved in FLT, (b) shows an iso-\(s\) line and an iso-\(\phi\) line of the manifold.
Expand the differential $X_R^{u/s}(s + ds, \phi + d\phi) - X_R^{u/s}(s, \phi)$ with $ds$ and $d\phi$,

$$
- R B_R / B_0 d\phi \text{ (simply along the field line, see Eqs. (29) )}
$$

$$
\frac{R B_R}{B_0} \frac{\partial X_R^{u/s}}{\partial s} \frac{ds}{d\phi} + \frac{\partial X_R^{u/s}}{\partial \phi} (\text{divide both sides by } d\phi)
$$

$$
= \frac{R B_R}{B_0} \frac{\partial X_R^{u/s}}{\partial s} \frac{ds}{d\phi} + \frac{\partial X_R^{u/s}}{\partial \phi} \overset{\text{Eq. (82)}}{=}
$$

$$
\pm \sqrt{\left( \frac{R B_R}{B_0} \frac{\partial X_R^{u/s}}{\partial s} \frac{ds}{d\phi} \right)^2 + \left( \frac{R B_R}{B_0} \frac{\partial X_R^{u/s}}{\partial \phi} \right)^2} = \frac{\partial X_R^{u/s}}{ds}.
$$

The $Z$ component $\partial X_R^{u/s} / ds$ is similar. □

**APPENDIX B. COMPARISON WITH ABDULLAEV’S WORK**

Abdullaev firstly transforms the standard cylindrical coordinates $(R, Z)$ to the canonically conjugated variables $(z, p_z)$:

$$
z = \frac{Z}{R_0}, \quad p_z = \frac{A_Z}{B_0 R_0}, \quad \psi_p = - \frac{R A_\phi}{B_0 R_0} \quad (1) \text{ in [29]}
$$

where $B_0$ is the toroidal magnetic field strength at the major radius $R_0$, and $A = (A_R = 0, A_Z, A_\phi)$ denotes the vector potential of magnetic field. The normalized poloidal flux $\psi_p$ serves as the Hamiltonian. However, this transform is not always viable when the magnetic field is non-axisymmetric (the definition of $\psi_p$ requires a closed flux surface diffeomorphic to $\mathbb{T}^2$), which limits the application of Abdullaev’s method to those near-integrable systems. After that, the FLT equations are presented in the following Hamiltonian form:

$$
\frac{dz}{d\phi} = \frac{\partial \psi_p}{\partial p_z}, \quad \frac{dp_z}{d\phi} = - \frac{\partial \psi_p}{\partial z} \quad (2) \text{ in [29]}
$$

A non-axisymmetric magnetic perturbation changes the original axisymmetric poloidal flux $\psi_p^{(0)}$ to:

$$
\psi_p = \psi_p^{(0)} (z, p_z) + \epsilon \psi_p^{(1)} (z, p_z, \phi), \quad (6) \text{ in [29]}
$$

where $\epsilon$ is a dimensionless perturbation parameter dictating the relative strength of perturbation, and $\psi_p^{(1)} (z, p_z, \phi)$ denotes the perturbation magnetic flux. Such a denotation implies an additional numeric step to transform the perturbation field $B_{\text{pert}}(R, Z, \phi)$ to $\psi_p^{(1)} (z, p_z, \phi)$. By comparison, $B = B_0(R, Z) + B_{\text{pert}}(R, Z, \phi)$ is taken as a whole in our theory, without need to distinguish the perturbation field from the field to be perturbed.

The Poincaré map in [29] is defined to be a full poloidal turn map, with the Poincaré section $\Sigma_z$ consisting of two segments of the $z$- and $\eta$-axes. $(\xi, \eta)$ is a rectangular coordinate system centered at the X-point (see Fig. 1 in [29]). Note that $\psi_p$ varies along the $\xi$-axis. Under the perturbation $\psi_p^{(1)}$, the Poincaré map $(\phi_k, \psi_k) \mapsto (\phi_{k+1}, \psi_{k+1})$ has the following classical general form:

$$
\psi_{k+1} = \psi_k + \epsilon \frac{\partial P (\phi_k \pm \pi q (\psi_k); \psi_{k+1})}{\partial \phi_k}
$$

$$
\phi_{k+1} = \phi_k + \epsilon \frac{\partial P (\phi_k \pm \pi q (\psi_k); \psi_{k+1})}{\partial \psi_{k+1}} \pm \pi \left[ q (\psi_k) + q (\psi_{k+1}) \right]
$$

(8) in [29]
where the upper and lower signs correspond to the $\phi$-increasing and $\phi$-decreasing Poincaré maps, respectively. $P(\phi; \psi)$ is an integral of $\psi^{(1)}_p$ along the unperturbed trajectory, a.k.a. the Poincaré integral,

$$P(\phi; \psi) = \int_{-\pi q(\psi)}^{\pi q(\psi)} \psi^{(1)}_p (z (\phi'; \psi), p_z (\phi'; \psi), \phi' + \phi) \, d\phi',$$

(9) in [29]

Be careful that the equality of Eqs. (8) only holds when $\epsilon$ is infinitesimal, otherwise we suggest to use $\approx$ instead of $=$ in Eqs. (8) or add a remainder to omit the high order terms for correctness. Eqs. (8) are the basis for many other equations in [29], which probably also needs to change to $\approx$ when applied to a realistic $\epsilon$.

Let $\psi_{k+1} = 0$ at $\phi_{k+1} \to \pm \infty$ in Eqs. (8), then Abdullaev immediately obtains the following implicit form of the subset of these two manifolds on $\Sigma_i$:

$$F^{(s,u)}(\phi, \psi) = \psi + \epsilon \frac{\partial}{\partial \phi} P(\phi \pm \pi q(\psi); 0) = 0,$$

(59, 60) in [29]

where the upper and lower signs correspond to the stable and unstable manifolds, respectively.

For the points of these two manifolds not on $\Sigma_i$, they are mapped to $\Sigma_i$ along the field lines by Eqs. (36) in [29]. Under the assumption that the magnetic perturbation at the X-point is small and can be neglected, $a_\alpha, a_\eta$ coefficient in Eqs. (36) are omitted to obtain the following simplified implicit manifold expression:

$$F^{(s,u)}(\phi, x, y) = \gamma xy + \epsilon \frac{\partial}{\partial \phi} P \left( \phi \pm \frac{1}{2 \gamma} \ln \frac{Q}{\gamma \vert x \vert} - \frac{1}{2 \gamma} \ln \frac{\bar{x}}{y}; 0 \right) = 0,$$

(70) in [29]

We note that $(x, y)$ in [29] are not the standard cylindrical coordinates $(R, Z)$, but defined by

$$x = \frac{a_\zeta + b_\eta}{\sqrt{2 \gamma}}, \quad y = \frac{-a_\zeta + b_\eta}{\sqrt{2 \gamma}}$$

part of (35) in [29]

where $a, b$ come from the second order term of the expansion of $h$ in the $(\xi, \eta, \phi)$ coordinate system.

$$h(\xi, \eta, \phi) \equiv \psi_p (z, p_z, \phi) - \psi^{(0)}_p (z, p_z)$$

$$\approx \epsilon \left[ a_\xi (\phi) \xi + a_\eta (\phi) \eta \right] - \frac{a^2}{2} \xi^2 + \frac{\beta^2}{2} \eta^2$$

(27) in [29]

We remind that $x$- and $y$-axes are not necessarily perpendicular. The slope of $x$-axis in $(\eta, \xi)$ plane is $\beta / a (y = (-a_\zeta + b_\eta) / \sqrt{2 \gamma} / 0$ on x-axis), while that of $y$-axis is $-\beta / a (x = (a_\zeta + b_\eta) / \sqrt{2 \gamma} / 0$ on y-axis). The product of their slopes equals $-\beta^2 / a^2$, which does not necessarily equal $-1$. To be honest, the $(x, y)$ coordinates adopted in [29] are not as convenient as ours, the standard cylindrical coordinates $(R, Z)$.

References


In this paper, we consider the growth of invariant manifolds in dynamical systems. The growth dynamics can be described using the parameterization method, which allows for the efficient computation of invariant manifolds in complex systems [1]. This technique is particularly useful in the context of plasma physics, where invariant manifolds play a crucial role in understanding transport phenomena [2].

The parameterization method is based on the use of normally hyperbolic invariant manifolds, which are central objects in the study of dynamical systems [3]. As shown in [4], the growth of these manifolds can be analyzed using sophisticated mathematical tools, such as the theory of normally hyperbolic invariant manifolds in dynamical systems [5].

In the context of plasma physics, the growth of invariant manifolds has been extensively studied, with many applications to the study of transport in tokamaks and other plasma devices [6]. The growth dynamics of invariant manifolds can provide insights into the transport properties of plasma systems [7].

The parameterization method has been successfully applied to a wide range of problems, including the study of transport in tokamaks, where it has been used to analyze the structure of island divertor and its impact on the divertor heat flux distribution in Wendelstein 7-X [8]. This approach has also been applied to the study of transport in other plasma devices, such as stellarators, where it has been used to analyze the structure of the plasma boundary and its impact on the plasma performance [9].

The parameterization method is particularly useful in the context of plasma physics, where it allows for the efficient computation of invariant manifolds in complex systems. In this paper, we have shown how the parameterization method can be used to study the growth of invariant manifolds in dynamical systems, and we have demonstrated the applicability of this approach to the study of transport in tokamaks and other plasma devices.
Invariant Manifold Growth


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