Results on fuzzy sets

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Abstract
This paper discusses some properties fuzzy sets.

Keywords: Kuratowski convergence

1. Introduction

Rojas-Medar and Román-Flores introduced the Kuratowski convergence of a sequence of sets in a metric space.

Let \((X, d)\) be a metric space. Let \(C\) be a set in \(X\) and \(\{C_n\}\) a sequence of sets in \(X\). \(\{C_n\}\) is said to **Kuratowski converge** to \(C\) according to \((X, d)\), if

\[
C = \liminf_{n \to \infty} C_n = \limsup_{n \to \infty} C_n,
\]

where

\[
\liminf_{n \to \infty} C_n = \{x \in X : x = \lim_{n \to \infty} x_n, x_n \in C_n\},
\]

\[
\limsup_{n \to \infty} C_n = \{x \in X : x = \lim_{j \to \infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} C_m.
\]

In this case, we’ll write \(C = \lim_{n \to \infty}^{(K)} C_n\) according to \((X, d)\). If there is no confusion, we will not emphasize the metric space \((X, d)\) and write \(\{C_n\}\) **Kuratowski converges** to \(C\) or \(C = \lim_{n \to \infty}^{(K)} C_n\) for simplicity.

Project supported by Natural Science Foundation of Fujian Province of China(No. 2020J01706)

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Lemma 1.1. (Lemma 2.1 in [1]) Let \((X, d)\) be a metric space, and \(C_n, n = 1, 2, \ldots\) be a sequence of sets in \(X\). Suppose that \(x \in X\). Then

(i) \(x \in \liminf_{n \to \infty} C_n\) if and only if \(\lim_{n \to \infty} d(x, C_n) = 0\),

(ii) \(x \in \limsup_{n \to \infty} C_n\) if and only if there is a subsequence \(\{C_{n_k}\}\) of \(C_n\) such that \(\lim_{k \to \infty} d(x, C_{n_k}) = 0\).

Proof. Here we give a detailed proof. Readers who think that this conclusion is obvious can skip this proof.

(i) Assume that \(x \in \liminf_{n \to \infty} C_n\). Then there is a sequence \(\{x_n, n = 1, 2, \ldots\}\) in \(X\) such that \(x_n \in C_n\) for \(n = 1, 2, \ldots\) and \(\lim_{n \to \infty} d(x, x_n) = 0\). Since \(d(x, C_n) \leq d(x, x_n)\), thus \(\lim_{n \to \infty} d(x, C_n) = 0\).

Conversely, assume that \(\lim_{n \to \infty} d(x, C_n) = 0\). For each \(n = 1, 2, \ldots\), we can choose an \(x_n\) in \(C_n\) such that \(d(x, x_n) \leq d(x, C_n) + 1/n\). Hence \(\lim_{n \to \infty} d(x, x_n) = 0\). So \(x \in \liminf_{n \to \infty} C_n\).

(ii) Assume that \(x \in \limsup_{n \to \infty} C_n\). Then there is a subsequence \(\{C_{n_k}\}\) of \(\{C_n\}\) and \(x_{n_k} \in C_{n_k}\) for \(k = 1, 2, \ldots\) such that \(\lim_{k \to \infty} d(x, x_{n_k}) = 0\). Since \(d(x, C_{n_k}) \leq d(x, x_{n_k})\), thus \(\lim_{k \to \infty} d(x, C_{n_k}) = 0\).

Conversely, assume that there is a subsequence \(\{C_{n_k}\}\) of \(\{C_n\}\) such that \(\lim_{k \to \infty} d(x, C_{n_k}) = 0\). For each \(k = 1, 2, \ldots\), we can choose an \(x_{n_k}\) in \(C_{n_k}\) such that \(d(x, x_{n_k}) \leq d(x, C_{n_k}) + 1/k\). Hence \(\lim_{k \to \infty} d(x, x_{n_k}) = 0\). So \(x \in \limsup_{n \to \infty} C_n\).

\[\square\]