A Statistical Fields Theory underlying the Thermodynamics of Ricci Flow and Gravity

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The paper proposes a statistical fields theory of quantum reference frame underlying the Perelman’s analogies between his formalism of the Ricci flow and the thermodynamics. The theory is based on a $d = 4 - \epsilon$ quantum non-linear sigma model (NLSM), interpreted as a quantum reference frame system which a to-be-studied quantum system is relative to. The statistic physics and thermodynamics of the quantum frame fields is studied by the density matrix of them obtained by the Gaussian approximation quantization. The induced Ricci flow of the frame fields and the Ricci-DeTurck flow of the frame fields associated with the density matrix is deduced. In this framework, the diffeomorphism anomaly of the theory has deep thermodynamic interpretation. The trace anomaly is related to a Shannon entropy in terms of the density matrix, which monotonically flows and achieves its maximal value at the flow limit, called the Gradient Shrinking Ricci Soliton (GSRS). A relative Shannon entropy w.r.t. the maximal entropy gives a statistical interpretation to Perelman’s partition function, which is also monotonic and giving an analogous H-theorem to the statistical frame fields system. We find that a temporal static 3-space of the GSRS spacetime is in a thermal equilibrium state, and Perelman’s analogies between his formalism and the thermodynamics of the frame fields in equilibrium can be explicitly given in the framework. As a possible underlying microscopic theory of the gravitational system, the theory is also applied to understand the thermodynamics of the Schwarzschild black hole. The cosmological constant in the effective theory of gravity at cosmic scale is also briefly given.

I. INTRODUCTION

Recent works [1, 2] show possible relations between Perelman’s formalism of the Ricci flow and some fundamental problems in quantum spacetime and quantum gravity, for instance, the trace anomaly and the cosmological constant problem. Perelman’s seminal works (the section-5 of [3]) and further development by Li [4, 5] also suggest deep relations between the Ricci flow and the thermodynamics system, not only the irreversible non-equilibrium but also the thermal equilibrium thermodynamics of certain underlying microscopic system. In [3] Perelman also declared a partition function and his functionals without specifying what the underlying microscopic ensemble really are (in physics). So far it is not clear whether the beautiful thermodynamic analogies are physical or pure coincidences. On the other hand, inspired by the surprising analogies between the black hole and thermodynamics system, it is generally believed the existence of temperature and entropy of a black hole. Works along this line also showed, in many aspects, the gravitational system would be profoundly related to thermodynamics system (see recent review [6] and references therein), it is generally conjectured that there would exist certain underlying statistical theory for the underlying microscopic quantum degrees of freedom of gravity. It gradually becomes one of the touchstones for a quantum gravity.

The motivations of the paper are, firstly, to propose an underlying statistical fields theory for Perelman’s seminal thermodynamics analogies of his formalism of the Ricci flow, and secondly, for understanding the possible microscopic origin of the spacetime thermodynamics especially the Schwarzschild black hole. We hope the paper could push forward the understanding to the possible interplay of the mysterious Perelman’s formalism of Ricci flow and the quantum spacetime and gravity. To our knowledge, several tentative works have been devoted to the goal, see e.g. [7–10], but frankly speaking, the physical picture underlying the Ricci flow is not fully clear, if a fundamental physical theory underlying the Ricci flow and a fundamental theory of quantum spacetime is lacking.

Based on our previous works [1, 2, 11–13] on the quantum reference frame and its relation to Perelman’s formalism of the Ricci flow, we propose a statistical fields theory of the quantum reference frame as a possible underlying theory of Perelman’s seminal analogies between his geometric functionals and the thermodynamic functions. In section II, we review the theory of quantum reference frame based on a $d = 4 - \epsilon$ quantum non-linear sigma model, at the Gaussian approximation quantization, we obtain a density matrix of the frame fields system as a physical foundation to the statistical interpretation of the theory. The induced Ricci flow of the frame fields and the Ricci-DeTurck flow of the frame fields associated with the density matrix is deduced. In section III, we discuss the diffeomorphism and related

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trace anomaly of the quantum frame fields theory and its profound implications to the irreversible non-equilibrium thermodynamics of the frame fields, for instance, the statistical entropy and an analogous H-theorem of the frame fields, and the effective gravity theory at cosmic scale (especially the emergence of the cosmological constant). In section IV, the thermal equilibrium state of the frame fields as a flow limit configuration (the Gradient Shrinking Ricci Soliton) is discussed, in which the density matrix recovers the thermal equilibrium canonical ensemble density. This section gives a physical foundation to Perelman’s seminal thermodynamic analogies. In section V, the framework is applied to give a possible microscopic understanding of the thermodynamics of the Schwarzschild black hole. Finally, we summarize the paper and give conclusions in the section VI.

II. QUANTUM REFERENCE FRAME

A. Definition

Reference frame is one of the most fundamental notions in physics. Any measurement in physics is performed or described, a reference frame has always been explicitly or implicitly used. In classical physics, the reference frame is idealizationally used via classical rulers and clocks to label the spacetime coordinates, which are classical, external, and rigid without any fluctuation. Even in the textbook quantum mechanics or quantum fields theory, the spacetime coordinates are still classical. But quantum principles tell us that all physical measuring instruments including the rulers and clocks are inescapably quantum fluctuating. Such idealizational and classical treatment of reference frame works not bad in quantum mechanics and quantum fields theory. To a large extent, this is due to the fact that the general coordinates transformation and gravitational effects are not seriously taken into account. Just as expected, when the quantum principles are seriously applied to the spacetime itself and gravitational phenomena, severe difficulties arise, e.g. information losses (non-unitary), diffeomorphism anomaly and the cosmological constant problems, etc.

In this section, we define a quantum fields theory of reference frame as a starting point to study a quantum theory of spacetime and corresponding quantum gravity (by assuming the equivalence principle is still valid at the quantum level). In this framework, a to-be-studied quantum system described by a state $|\psi\rangle$ and the spacetime reference system by $|X\rangle$ are both quantum. The states of the whole system are given by an entangled state

$$|\psi[X]\rangle = \sum_{ij} \alpha_{ij} |\psi_i\rangle \otimes |X_j\rangle$$

in their direct product Hilbert space $\mathcal{H}_\psi \otimes \mathcal{H}_X$. The state $|\psi[X]\rangle$ describes the physical system being in state $|\psi_i\rangle$ w.r.t. the state $|X_j\rangle$ of the quantum reference system. The state replaces and generalizes the textbook quantum description of the state $|\psi(x)\rangle$ w.r.t. a parameter $x$ in an classical inertial frame (in quantum mechanics $x$ is the Newtonian time, in quantum fields theory $x_a$ are the Minkowskian spacetime). The entangled state $|\psi[X]\rangle$ is inseparable, so that the state can only be interpreted in a relational manner, i.e. the individual state $|\psi\rangle$ has no absolute meaning without being reference to $|X\rangle$ entangled to it. Since the state of reference $|X\rangle$ is also subject to quantum fluctuation, so mathematically speaking, the state $|\psi(X)\rangle$ can be seen as the state $|\psi(x)\rangle$ with a smeared spacetime coordinates, instead of the textbook state $|\psi(x)\rangle$ with a definite and classical spacetime coordinates. The state $|\psi(X)\rangle$ could recover the textbook state $|\psi(x)\rangle$ only when the quantum fluctuation of the reference system is small enough and hence can be ignored. More precise, the 2nd order central moment (even higher order central moments) fluctuations of the spacetime coordinate $\langle \delta X^2 \rangle$ (the variance) can be ignored compared with its 1st order moment of quadratic distance $\langle \Delta X \rangle^2$ (squared mean), where (…) represents the quantum expectation value by the state of the reference system $|X\rangle$.

To find the state $|X\rangle$ of the quantum reference system, a quantum theory of the reference frame must be introduced. From the mathematical viewing point, to defined a D-dimensional manifolds we need to construct a non-linear differentiable mapping $X$ from a local coordinate patch $x \in \mathbb{R}^d$ to a D-manifolds $M^D$. The mapping in physics is usually realized by a kind of fields theory, the non-linear sigma model (NLSM) [14–21]

$$S[X] = \frac{1}{2} \int d^d x g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a},$$

where $\lambda$ is a constant with dimension of energy density $[E^{-d}]$. $x_a (a = 0, 1, 2, \ldots, d-1)$, with dimension length $[L]$, is called the base space in NLSM’s terminology, representing the coordinates of the local patch. For the reason that a quantum fields theory must be formulated in a classical inertial frame, i.e. flat Minkowskian or Euclidean spacetime, so the base space is considered flat. Without loss of generality, we consider the base space as the Euclidean one, i.e. $x \in \mathbb{R}^d$ which is better defined when one tries to quantize the theory.
The differential mapping $X_\mu(x)$, $\mu = 0, 1, 2, \ldots, D - 1$, with dimensional length $[L]$, is the coordinates of a general Riemannian or Lorentzian manifolds $M^D$ (depending on the boundary condition) with curved metric $g_{\mu\nu}$, called the target space in NLSM's terminology. We will work with the real-defined coordinates for the target spacetime, and the Wick rotated version has been included into the general coordinates transformation of the time component. In the language of quantum fields theory, $X_\mu(x)$ or $X^\mu(x) = g^{\mu\nu}X_\nu(x)$ are the real scalar frame fields.

From the physical viewing point, the reference frame fields can be interpreted as a physical coordinates system by using particle/fields signals, for instance, a multi-wire proportional chamber that measuring coordinates of an event in a lab. To build a coordinates system, first we need to orient, align and order the array of the multi-wires with the reference to the wall of the lab in a lab. To build a coordinates system, first we need to orient, align and order the array of the multi-wires with the reference to the wall of the lab in NLSM’s terminology. We will work with the real-defined coordinates for the target spacetime, and the reference to the wall of the lab in a lab. To build a coordinates system, first we need to orient, align and order the array of the multi-wires with the reference to the wall of the lab in NLSM’s terminology. We will work with the real-defined coordinates for the target spacetime, and the reference to the wall of the lab in NLSM’s terminology. We will work with the real-defined coordinates for the target spacetime, and the reference to the wall of the lab in NLSM’s terminology.

Similarly, one need to read an extra electron signal large, so that the intensity can be seen as a linear function of the coordinates of the lab's wall, number counting. Usually we could consider the electrons in the wires are free, and the field's intensity is not very large, so that the intensity can be seen as a linear function of the coordinates of the lab's wall, $X_\mu(x) = \sum_{a=1}^{3} e_\mu^a x_a$, $\mu = 1, 2, 3$, for instance, here $e_\mu^a = \delta_\mu^a$ is the intensity of the signals in each orthogonal direction. Meaning that when the direction $\mu$ is the lab's wall direction $a$, the intensity of the electron beam is 1, otherwise the intensity is 0. Similarly, one need to read an extra electron signal $X_0$ to know when the event happens, with the reference to the lab's clock $x_0$. Thus, the fields of these 3+1 electron signals can be given by

$$X_\mu(x) = \sum_{a=0}^{3} e_\mu^a x_a, \quad (\mu = 0, 1, 2, 3). \quad \text{(3)}$$

The intensity of the fields $e_\mu^a$ is in fact the vierbein, describing a mapping from the lab coordinate $x_a$ to the frame fields $X_\mu$.

When the event happens at a long distance beyond the lab's scale, for instance, at the scale of earth or solar system, we could imagine that to extrapolate the multi-wire chamber to such long distance scale still seems OK, only replacing the electrons beam in wire by the light beam. However, if the scale is much larger than the solar system, for instance, to the galaxy or cosmic scale, when the signal travels along such long distance and be read by an observer, we could imagine that the broadening of the light beam fields or other particle fields gradually becomes unignorable. More precisely, the 2nd (or higher) order central moment fluctuations of the frame fields signals can not be neglected, the distance of Riemannian spacetime as a quadratic form must be modified by the 2nd moment fluctuation or variance $\langle \delta X^2 \rangle$ of the coordinates

$$\langle (\Delta X)^2 \rangle = \langle \Delta X \rangle^2 + \langle \delta X^2 \rangle. \quad \text{(4)}$$

A local distance element in Riemannian spacetime is given by a local metric tensor at the point, so it is convenient to think of the location point $X$ being fixed, and interpreting the variance of the coordinate affect only the metric tensor $g_{\mu\nu}$ at the location point. As a consequence, the expectation value of a metric tensor $g_{\mu\nu}$ is corrected by the 2nd central moment quantum fluctuation of the frame fields

$$\langle g_{\mu\nu} \rangle = \left\langle \frac{\partial X_\mu}{\partial x_a} \frac{\partial X_\nu}{\partial x_a} \right\rangle = \left\langle \frac{\partial X_\mu}{\partial x_a} \right\rangle \left\langle \frac{\partial X_\nu}{\partial x_a} \right\rangle + \left\langle \frac{\partial^2}{\partial x_a^2} (\Delta X_\mu \Delta X_\nu) \right\rangle = g_{\mu\nu}^{(1)}(X) + \delta g_{\mu\nu}^{(2)}(X), \quad \text{(5)}$$

where

$$g_{\mu\nu}^{(1)}(X) = \left\langle \frac{\partial X_\mu}{\partial x_a} \right\rangle \left\langle \frac{\partial X_\nu}{\partial x_a} \right\rangle = \langle e_\mu^a \rangle \langle e_\nu^a \rangle \quad \text{(6)}$$

is the 1st order moment (mean value) contribution to the classical Riemannian spacetime. For the 2nd order central moment $\delta g_{\mu\nu}^{(2)}$ (variance), even if the spacetime is classically non-curved, the expectation value of the metric generally tends to be curved due to the existence of the 2nd moment fluctuation, the longer the distance scale the more important the broadening of the frame fields, making the spacetime geometry gradually deform and flow at long distance scale.

Since the classical solution of the frame fields (3) given by the vierbein satisfying the classical equation of motion of the NLSM, it is a frame fields interpretation of NLSM in a lab: the base space of NLSM is interpreted as a starting reference by the lab’s wall and clock, the frame fields $X(x)$ on the lab are the physical instruments measuring the spacetime coordinates. In this interpretation we consider $d = 4 - \epsilon$, $0 < \epsilon \ll 1$ in (2) and $D = 4$ is the least number of the frame fields.

There are several reason why $d$ is not precise but very close to 4 in the quantum frame fields interpretation of NLSM. $d$ must be very close to 4, first, certainly at the scale of lab it is our common sense; Second if we consider the
entangled system $\mathcal{H}_\psi \otimes \mathcal{H}_X$ between the to-be-studied physical system and the reference frame fields system. Without loss of generality, we could take a scalar field $\psi$ as the to-be-studied (matter) system, which shares the base space with the frame fields, the total action of the two entangled system is a direct sum of each system

$$S[\psi, X] = \int d^d x \left[ \frac{1}{2} \frac{\partial \psi}{\partial x_a} \frac{\partial \psi}{\partial x_a} - V(\psi) + \frac{1}{2} \lambda g_{\mu \nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right],$$

(7)

where $V(\psi)$ is some potential of the $\psi$ fields. Since both $\psi$ field and the frame fields $X$ share the same base space, here they are described w.r.t. the base space as usual. If we interpret the frame fields as the physical spacetime coordinates, the coordinate of $\psi$ field must be transformed from $x$ to $X$. At the semi-classical level, or 1st moment approximation when the fluctuation of $X$ can be ignored, it is simply a coordinates transformation

$$S[\psi, X] \approx S[\psi(X)] = \int d^d X \sqrt{\left| \det g^{(1)} \right|} \left[ \frac{1}{4} \left\langle g^{(1)}_{\mu \nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right\rangle \left( \frac{1}{2} g^{(1)\mu \nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} + 2\lambda \right) - V(\psi) \right],$$

(8)

in which $^{(1)}$ stands for the 1st moment approximation, and $\frac{1}{4} \left\langle g^{(1)}_{\mu \nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right\rangle = \frac{1}{4} \left\langle g^{(1)\mu \nu} g^{(1)\mu \nu} \right\rangle = \frac{1}{4} D = 1$ has been used. It is easy to see, at the semi-classical level, i.e. only consider the 1st moment of $X$ while 2nd moment fluctuations are ignored, the (classical) coordinates transformation reproduces the scalar field action in general coordinates $X$ up to a constant $2\lambda$, and the derivative $\frac{\partial}{\partial x_a}$ is replaced by the functional derivative $\frac{\delta}{\delta X^a}$. $\sqrt{\left| \det g^{(1)} \right|}$ is the Jacobian determinant of the coordinate transformation, note that the determinant requires the coordinates transformation matrix a square matrix, so at semi-classical level $d$ must be very close to $D = 4$, which is not necessarily true beyond the semi-classical level, when the 2nd moment quantum fluctuations are important. For instance, since $d$ is a parameter but an observable in the theory, it could even not necessary be an integer but effectively fractal at the quantum level.

$d$ is not precisely 4 is for the quantum and topological reasons. To investigate this, we note that quantization depends on the homotopy group $\pi_d(M^D)$ of the mapping $X(x) : \mathbb{R}^d \to M^D$. If we consider $M^D$ a compact $S^D$ for simplicity, the group is trivial for all $d < D = 4$, in other words, when $d < 4$ the mapping $X(x)$ will be free from any unphysical singularities for topological reason, in this situation, the target spacetime is always mathematically well-defined. However, the situation $d = 4$ is a little subtle, since $\pi_4(S^4) = \mathbb{Z}$ is non-trivial, the mapping may meet intrinsic topological obstacle and become singular. When the quantum fluctuation is taken into account, this situation can not be avoided, and by its RG flow the spacetime is possibly deformed into intrinsic singularities making the theory ill-defined and non-renormalizable (RG flow not converge). So at the quantum level, $d = 4$ should be not precisely, we have to assume $d = 4 - \epsilon$ when the quantum principle is applying, while at the classical or semi-classical level, considering $d = 4$ has no problem. If we consider $M^D$ noncompact, e.g., $\pi_d(M^{4+1})$ may be trivial for $d = 4$, only in this situation, the dimension of the base space could be exactly 4. The above argument is different from the conventional simple power counting argument, which claims the NLSM is perturbative non-renormalizable when $d > 2$, but it is not necessarily the case, some numerical calculations also support $d = 3$ and $d = 4 - \epsilon$ are non-perturbative renormalizable and well-defined at the quantum level.

B. Beyond the Semi-Classical Level: Gaussian Approximation

Going beyond the semi-classical or 1st order moment approximation, we need to quantize the theory at least at the next leading order. If we consider the 2nd order central moment quantum fluctuation are the most important next leading order contribution (compared with higher order moment), we call it the Gaussian approximation or 2nd order central moment approximation, while the higher order moment are all called non-Gaussian fluctuations which may be important near local singularities of the spacetime when local phase transition happens, although the intrinsic global singularity can be avoided by guaranteeing the global homotopy group trivial.

At the Gaussian approximation, $\delta g^{(2)}_{\mu \nu}$ can be given by a perturbative one-loop calculation [18, 19] of the NLSM when it is relatively small compared with $g^{(1)}_{\mu \nu}$

$$\delta g^{(2)}_{\mu \nu}(X) = \frac{R^{(1)}_{\mu \nu}(X)}{32\pi^2 \lambda} \delta k^2,$$

(9)

where $R^{(1)}_{\mu \nu}$ is the Ricci curvature given by 1st order metric $g^{(1)}_{\mu \nu}$, $k^2$ is the cutoff energy scale. The validity of the perturbation calculation $R^{(1)} \delta k^2 \ll \lambda$ is the validity of the Gaussian approximation, which can be seen as follows.
It will be shown in later section that \( \lambda \) is nothing but the critical density \( \rho_c \) of the universe, \( \lambda \sim O(H_0^2/G) \), \( H_0 \) the current Hubble’s parameter, \( G \) the Newton’s constant. Thus for our concern of pure gravity in which matter is ignored, the condition \( R^{(1)} k^2 \ll \lambda \) is equivalent to \( \delta k^2 \ll 1/G \) which is reliable except for some local singularities are developed when the Gaussian approximation is failed.

The equation (9) is nothing but a RG equation or known as the Ricci flow equation (some reviews see e.g. [22–24])

\[
\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu},
\]  

with flow parameter \( \delta t = -\frac{1}{\delta X^k \delta k^2} \) having dimension of length squared \([L^2]\), which continuously deform the spacetime metric driven by its Ricci curvature.

For the Ricci curvature is non-linear for the metric, the Ricci flow equation is a non-linear version of a heat equation for the metric, and flow along \( t \) introduces an averaging or coarse-graining process to the intrinsic non-linear gravitational system which is highly non-trivial [25–29]. In general, if the flow is free from local singularities there exists long flow-time solution in \( t \in (-\infty, 0) \), which is often called ancient solution in mathematical literature. This range of the \( t \)-parameter corresponds to \( k \in (0, \infty) \), that is from \( t = -\infty \) i.e. the short distance (high energy) UV scale \( k = \infty \) forwardly to \( t = 0 \) i.e. the long distance (low energy) IR scale \( k = 0 \). The metric at certain scale \( t \) is given by being averaged out the shorter distance details which produces an effective correction to the metric at that scale. So along \( t \), the manifolds loss its information in shorter distance, thus the flow is irreversible, i.e. generally having no backwards solution, which is the underlying reason for the non-unitary and existence of entropy of a spacetime.

As it is shown in (4), (5), the 2nd order moment fluctuation modifies the local (quadratic) distance of the spacetime, so the flow is non-isometry. This is an important feature worth stressing, which is the underlying reason for the anomaly. The non-isometry is not important for its topology, so along \( t \), the flow preserves the topology of the spacetime but its local metric, shape and size (volume) changes. There also exists a very special solution of the Ricci flow called Ricci Soliton, which only changes the local volume while keeps its local shape. The Ricci Soliton, and its generalized version, the Gradient Ricci Soliton, as the flow limits, are the generalization of the notion of fixed point in the sense of RG flow. The Ricci Soliton is an important notion for understanding the gravity at cosmic scale and studying the the thermodynamics of the Ricci flow at equilibrium.

The Ricci flow was initially introduced in 1980s by Friedan [15, 16] in \( d = 2 + \epsilon \) NLSM and independently by Hamilton in mathematics [30, 31]. The main motivation of introducing it from the mathematical point of view is to classify manifolds, a specific goals is to proof the Poincare conjecture. Hamilton used it as a useful tool to gradually deform a manifolds into a more and more “simple and good” manifolds whose topology can be readily recognized for some simple cases. A general realization of the program is achieved by Perelman at around 2003 [3, 32, 33], who introduced several monotonic functionals to successfully deal with the local singularities which may be developed in more general cases. The Ricci flow approach is not only powerful to the compact geometry (as Hamilton’s and Perelman’s seminal works had shown) but also to the non-compact geometry [34–36] (short flow-time existence and the uniqueness of the solution).

C. The Wavefunction and Density Matrix at the Gaussian Approximation

So far we have not explicitly defined the quantum state of the reference frame \( |X\rangle \) in (1). In fact, the previous (2nd order) results e.g. (5), (9) and hence the Ricci flow (10) can also equivalently be given by the expectation value \( \langle O \rangle = \langle X|O|X\rangle \) via explicitly writing down the wavefunction \( \Psi(X) \) of the frame fields at the Gaussian approximation. Note that at the semi-classical level, the frame fields \( X \) is a delta-distribution and peaks at its mean value, and further more, the action of the NLSM seem like a collection of harmonic oscillators, thus at the Gaussian approximation level, finite Gaussian width/2nd moment fluctuation of \( X \) must be introduced. Thus the fundamental solution of the wave function(al) takes the Gaussian form

\[
\Psi[X^\mu(x)] = \frac{\det(\sigma_{\mu\nu})^{1/4}}{\sqrt{X(2\pi)^D/4}} \exp\left[-\frac{1}{4} X^\mu(x)\sigma_{\mu\nu}X^\nu(x)\right],
\]

where the covariant matrix \( \sigma_{\mu\nu}(x) \), playing the role of the Gaussian width, is the inverse of the 2nd order central moment fluctuations of the frame fields at point \( x \)

\[
\sigma_{\mu\nu}(x) = \frac{1}{\sigma^{\mu\nu}(x)} = \frac{1}{\langle \delta X^\mu(x)\delta X^\nu(x)\rangle},
\]

which is also given by perturbative one-loop calculation up to a diffeomorphism of \( X \).
We can also define a dimensionless density matrix corresponding to the fundamental solution of the wavefunction

$$u[X^\mu(x)] = \Psi^*(X)\Psi(X) = \frac{\det (\sigma_{\mu\nu})^{1/2}}{\lambda(2\pi)^{D/2}} \exp \left[ -\frac{1}{2} X^\mu(x)\sigma_{\mu\nu} X^\nu(x) \right],$$

and $\frac{\det (\sigma_{\mu\nu})^{1/2}}{\lambda(2\pi)^{D/2}}$ is a normalization parameter, so that

$$\lambda \int d^D X \Psi^*(X)\Psi(X) = \lambda \int d^D X u(X) = 1,$$

then the expectation values $\langle O \rangle$ can be understood as $\lambda \int d^D X uO$.

Under a diffeomorphism of the metric, the transformation of $u(X)$ is given by a diffeomorphism of the covariant matrix ($h$ is certain function)

$$\sigma_{\mu\nu} \to \hat{\sigma}_{\mu\nu} = \sigma_{\mu\nu} + \nabla_\mu \nabla_\nu h.$$

So there exists an arbitrariness in the density $u(X)$ for different choices of a diffeomorphism/gauge.

According to the statistical interpretation of wavefunction with the normalization condition (14), $u(X^0, X^1, X^2, X^3)$ describes the probability density that finding these frame particles in the volume $d^D X$. Since the frame fields, on the one hand, represent the frame particles, on the other hand, also represent the spacetime they live in, so we could use the convention that the frame particle is represented by the contravariant index $X^\mu$, and the spacetime they live in is represented by the covariant index $X_\mu$. As the spacetime $X$ flows along $t$, the volume $\Delta V_t$ in which the density is averaged also flows, so the density at the corresponding scale must be coarse-grained. If we consider the volume of the lab, i.e. the base space, is rigid and fixed to be $\lambda \int d^D x = 1$, so

$$u(X(x), t) = \frac{d^4 x}{d^D X_t} = \lim_{\Delta V_t \to 0} \frac{1}{\Delta V_t} \int_{\Delta V_t} 1 \cdot d^4 x.$$

Thus the density $u(X, t)$ can be interpreted as a coarse-grained density at the scale $t$ w.r.t. a fine-grained unit density in the lab at UV $t \to -\infty$.

In this sense, the coarse-grained density $u(X, t)$ is in analogy with the Boltzmann’s distribution function, so it should satisfy an analogous irreversible Boltzmann’s equation, and giving rise to an analogous Boltzmann’s monotonic H-functional. In the following sections, we will deduce such equation and the functional of $u(X, t)$.

The coarse-grained density $u(X, t)$ has profound physical and geometric meaning, playing a central role in analyzing the statistic physics of the frame fields and generalizing the Riemannian manifolds to the density manifolds.

**D. Ricci-DeTurck Flow**

The eq. (16) can also be interpreted as a manifolds density from the geometric point of view. It is worth stressing that $u(X, t)$ associates a manifolds density to each point $X$, which is not equivalent to scaling the metric conformally by a factor, since in that case the integral measure of 4-volume or 3-volume in the expectation $\langle O \rangle = \lambda \int d^D X uO$ would scale by different powers. There are various useful generalizations of the Ricci curvature to the density manifolds, a widely accepted version is the Bakry-Emery generalization \[37\]

$$R_{\mu\nu} \to R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u,$$

which is also used in Perelman’s seminal paper. The density normalized Ricci curvature is bounded from below

$$R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u \geq \sigma_{\mu\nu},$$

if the density manifolds has finite volume.

Thus, we get the Ricci-DeTurck flow \[38\]

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2 (R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u),$$

which is equivalent to the standard Ricci flow equation (10) up to a diffeomorphism. Mathematically, the Ricci-DeTurck flow has the advantage that it turns out to be a gradient flow of some monotonic functionals introduced by Perelman, which have profound physical meanings.
The eq.(16) also gives a volume constraint to the fiducial spacetime (the lab), the coarse-grained density $u(X,t)$ cancels the flow of the volume element $d^DX_t$ or $\sqrt{|\det g_{\mu\nu}|}$, so

$$\frac{\partial}{\partial t} \left( u \sqrt{|\det g_{\mu\nu}|} \right) = 0.$$  \hspace{1cm} (20)

Together with the Ricci-DeTurck flow equation (19), we have the flow equation of the density

$$\frac{\partial u}{\partial t} = (R - \Delta_X) u,$$  \hspace{1cm} (21)

which is in analogy to the irreversible Boltzmann's equation for his distribution function. Note the minus sign in front of the Laplacian, it is a backwards heat-like equation. Naively speaking, the solution of the backwards heat flow will not exist. But we could also note that if one let the Ricci flow flows to certain IR scale $t_*$, and at $t_*$ one may then choose an appropriate $u(t_*) = u_0$ arbitrarily (up to a diffeomorphism gauge) and flows it backwards in $\tau = t_* - t$ to obtain a solution $u(\tau)$ of the backwards equation. Now since the flow is consider free from global singularities for the trivialness of homotopy group, we could simply choose $t_* = 0$, so we defined

$$\tau = -t = \frac{1}{64\pi^2 \lambda} k^2 \in (0, \infty).$$  \hspace{1cm} (22)

In this case, the density satisfies the heat-like equation

$$\frac{\partial u}{\partial \tau} = (\Delta_X - R) u,$$  \hspace{1cm} (23)

which does admit a solution along $\tau$, often called the conjugate heat equation in mathematical literature.

So far (23) together with (19) the mathematical problem of the Ricci flow of a Riemannian manifolds is transformed to a coupled equations

$$\begin{cases}
\frac{\partial g_{\mu\nu}}{\partial t} = -2 \left( R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u \right) \\
\frac{\partial u}{\partial \tau} = (\Delta_X - R) u \\
\frac{d\tau}{dt} = -1
\end{cases}$$  \hspace{1cm} (24)

and the pure Riemannian manifolds $(M^D, g)$ is generalized to a density manifolds $(M^D, g, u)$ \cite{39–41} with the constraint (14).

\section{III. The Anomaly and Its Implications}

\subsection{A. Diffeomorphism at the Quantum Level}

As is seen previous, the quantum fluctuation and hence the Ricci flow does not preserve the quadratic distance of a Riemannian geometry. The non-isometry of the quantum fluctuation induces a breakdown of diffeomorphism or general coordinate transformation at the quantum level, namely the anomaly. More precisely, here we consider functional quantization of the pure frame fields, the partition function is

$$Z(M^D) = \int [DX] \exp \left( -S[X] \right) = \int [DX] \exp \left( -\frac{1}{2} \lambda \int d^4x g^{\mu\nu} \partial_\mu X_\mu \partial_\nu X_\nu \right),$$  \hspace{1cm} (25)

where $M^D$ is the target Riemannian spacetime, and the base space can be either Euclidean and Minkowskian. Since considering the action or the volume element $d^4x \equiv d^4x \det e$ (det e is a Jacobian) does not pick any imaginary $i$ factor no matter the base space is in Minkowskian or Euclidean one, if one takes $dx_0^{(E)} \rightarrow idx_0^{(M)}$ then $\det e^{(E)} \rightarrow -i \det e^{(M)}$, so without loss of generality we use the Euclidean base spacetime in the following discussions, and remind that the result is the same for Minkowskian.

Note that a general coordinate transformation

$$X_\mu \rightarrow \tilde{X}_\mu = \frac{\partial \tilde{X}_\mu}{\partial X_\nu} X_\nu = e'_\mu X_\nu$$  \hspace{1cm} (26)

does not change the action $S[X] = S[\tilde{X}]$, but the measure of the functional integral changes

$$D\tilde{X} = \prod_x \prod_{\mu=0}^{D} d\tilde{X}_\mu(x) = \prod_x \epsilon_{\mu\nu\rho\sigma}^0 \epsilon^1_\mu c^2 e^3 dX_0(x) dX_1(x) dX_2(x) dX_3(x)$$

$$= \prod_x |\det e(x)| \prod_x \prod_{a=0}^{D} dX_a(x) = \left( \prod_x |\det e(x)| \right) D X,$$

where

$$\epsilon_{\mu\nu\rho\sigma}^0 \epsilon^1_\mu c^2 e^3 = |\det e^a_\mu| = \sqrt{|\det g_{\mu\nu}|}$$

is the Jacobian of the diffeomorphism. The Jacobian is nothing but a local relative volume element.

The fiducial volume does not change the action and hence the non-unitarity are valid not only for spacetime with Euclidean but also for the Lorentzian signature, the pure real contribution of the real-defined volume form (29) for both Euclidean and Lorentzian signature, the pure real contribution of the change, and consequently the non-invariance of the measure of the functional integral during the Ricci flow. Because the classical action (27) is purely due to the quantum fluctuation and Ricci flow of the frame fields which do not preserve the functional integral measure and change the spacetime volume at the quantum level. The diffeomorphism anomaly is purely due to the quantum fluctuation and Ricci flow of the frame fields which do not preserve the functional integral measure and change the spacetime volume at the quantum level. The quantum partition function is no longer invariant under the general coordinates transformation or diffeomorphism, but the

$$u(\tilde{X}) = \frac{d\mu(X_\mu)}{d\mu(X_\mu)} = |\det e^a_\mu| = \frac{1}{|\det e^a_\mu|},$$

Here the absolute value symbol of the determinant is because the density $u$ and the volume element are kept positive defined even in the Lorentz signature. Otherwise, for the Lorentz signature, there should introduce some extra imaginary factor $i$ into (30) to keep the condition (14).

In this case, if we parameterize a dimensionless solution $u$ of the conjugate heat equation as

$$u(\tilde{X}) = \frac{1}{\lambda(4\pi)^{D/2}} e^{-f(\tilde{X})},$$

then the partition function $Z(M^D)$ is transformed to

$$Z(\tilde{M}^D) = \int [D\tilde{X}] \exp \left( -S[\tilde{X}] \right) = \int \left( \prod_x |\det e| \right) [D X] \exp \left( -S[X] \right)$$

$$= \int \left( \prod_x e^{f + \frac{D}{2} \log(4\pi)} \right) [D X] \exp \left( -S[X] \right)$$

$$= \int [D X] \exp \left\{ -S[X] + \lambda \int d^4x \left[ f + \frac{D}{2} \log(4\pi) \right] \right\}$$

$$= \int [D X] \exp \left\{ -S[X] + \lambda \int M^D d^D X u \left[ f + \frac{D}{2} \log(4\pi) \right] \right\}.$$

Note that the change of the partition function

$$Z(\tilde{M}^D) = e^{N(\tilde{M}^D)} Z(M^D)$$

is nothing but a pure real Shannon entropy in terms of density matrix $u$

$$N(\tilde{M}^D) = \int_{\tilde{M}^D} d^D X u \left[ f + \frac{D}{2} \log(4\pi) \right] = - \int_{\tilde{M}^D} d^D X u \log u.$$

The classical action $S[X]$ is invariant under the general coordinates transformation or diffeomorphism, but the quantum partition function is no longer invariant under the general coordinates transformation or diffeomorphism, which is called diffeomorphism anomaly, meaning a breaking down of the diffeomorphism at the quantum level. The diffeomorphism anomaly is purely due to the quantum fluctuation and Ricci flow of the frame fields which do not preserve the functional integral measure and change the spacetime volume at the quantum level. The diffeomorphism anomaly has many profound consequences to the theory of quantum reference frame, e.g. non-unitarity, the trace anomaly, the notion of entropy, reversibility, and the cosmological constant.

The non-unitarity is indicated by the pure real anomaly term, which is also induced by the non-isometry or volume change, and consequently the non-invariance of the measure of the functional integral during the Ricci flow. Because of the real-defined volume form (29) for both Euclidean and Lorentzian signature, the pure real contribution of the anomaly and hence the non-unitarity are valid not only for spacetime with Euclidean but also for the Lorentzian
It is a rather general consequence of the Ricci flow of spacetime. Essentially speaking, the reason of the non-unitarity is because we have enlarged the Hilbert space of the reference frame, from a rigid classical frame to a fluctuating quantum frame. The non-unitarity implies the breakdown of the fundamental Schrödinger equation which is only valid on a classical time of inertial frame, the solution of which is in $H_\psi$. A fundamental equation playing the role of the Schrödinger equation, which can arbitrarily choose any (quantum) physical system as time or reference frame, must be replaced by a Wheeler-DeWitt-like equation in certain sense [11], the solution of which is insteadly in $H_\psi \otimes H_X$. In the fundamental equation, the quantum fluctuation of physical time and frame, more generally, a general physical coordinates system must break the unitarity. We know that in quantum fields theory on curved spacetime or accelerating frame, the vacuum states of the quantum fields in difference diffeomorphism equivalent coordinate systems are unitarily inequivalent. The Unruh effect is a well known example: accelerating observers in the vacuum will measure a thermal bath of particles. The Unruh effect shows us how a general coordinates transformation (from an inertial to an accelerating frame) leads to the non-unitary anomaly (particle creation and hence particle number non-conservation), and how the anomaly will relate to a thermodynamics system (thermal bath). In fact, like the Unruh effect, the Hawking effect [42] and all non-unitary particle creation effects in a curved spacetime or accelerating frame are related to the anomaly in a general covariant or gravitational system. All these implies that the diffeomorphism anomaly will have deep thermodynamic interpretation which is the central issue of the paper.

Without loss of generality, if we simply consider the under-transformed coordinates $X_\mu$ as the coordinates of the fiducial lab $x_\mu$ which can be treated as a classical parameter coordinates, in this situation the classical action of NLSM is just a topological invariant, i.e. half the dimension of the target spacetime

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$$\exp(-S_{cl}) = \exp \left(-\frac{1}{2} \lambda \int d^4x g^{\mu\nu} \partial_\mu x_\mu \partial_\nu x_\nu \right) = \exp \left(-\frac{1}{2} \lambda \int d^4x g^{\mu\nu} g_{\mu\nu} \right) = e^{-\frac{D}{2}}. \quad (34)$$

Thus the total partition function of the frame fields takes a simple form

$$Z(M^D) = e^{\lambda N(M^D) - \frac{D}{2}}. \quad (35)$$

B. The Trace Anomaly

The partition function now is non-invariance (32) under diffeomorphism at the quantum level, so if one deduces the stress tensor by $\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{|g|}} \frac{\delta \log Z}{\delta g^{\mu\nu}}$, its trace $\langle g^{\mu\nu} \rangle \langle T_{\mu\nu} \rangle$ is difference from $\langle T_{\mu}^{\mu} \rangle = \langle g^{\mu\nu} T_{\mu\nu} \rangle$

$$\langle \Delta T_{\mu}^{\mu} \rangle = \langle g^{\mu\nu} \rangle \langle T_{\mu\nu} \rangle - \langle g^{\mu\nu} T_{\mu\nu} \rangle \neq 0 \quad (36)$$

known as the trace anomaly. Cardy conjectured [43] that in a $d = 4$ theory, quantities like $\langle T_{\mu}^{\mu} \rangle$ may be a higher dimensional generalization of the monotonic Zamolodchikov’s $c$-function in $d = 2$ conformal theories, leading to a suggestion of the a-theorem [44] in $d = 4$ and other suggestions (e.g. [45, 46]). In the following subsections, we will show that the Shannon entropy $N$ and generalized $\tilde{N}$ are indeed monotonic, which may have more advantages, e.g. suitable for a noncompact target spacetime and for general $D$.

Note that the non-trivial anomalous $e^{\lambda N}$ term is nothing but the determinant of the inverse of the heat kernel solution of the conjugate heat equation (23) with the Laplacian $\Delta_L = 4\Delta_X - R$

$$e^{\lambda N(M^D)} = \lambda (4\pi t)^{D/2} \text{tr} e^{-t\Delta_L}, \quad (37)$$

which can be expanded [47, 48] by the DeWitt-Schwinger series at small $t$.

$$e^{\lambda N(M^D)} = \lambda (4\pi t)^{D/2} \text{tr} e^{-t\Delta_L} = \lambda \sum_{k=0}^{\infty} B_k t^{k/2} = \lambda \left( B_0 + B_2 t + B_4 t^2 + \ldots \right) \quad (t \to 0). \quad (38)$$

For $D = 4$ and harmonic coordinates, the first few coefficients are

$$B_0 = \int_{M^4} d^4x \sqrt{|g|} = \text{Vol}(M^4), \quad (39)$$

$$B_2 = -\int_{M^4} d^4x \sqrt{|g|} R, \quad (40)$$
\[ B_4 = \frac{4}{5} \int_{M^4} d^4x \sqrt{|g|} \left[ -R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{2} \Delta_X R + \frac{5}{8} R^2 \right], \tag{41} \]

in which \( B_0, B_2 \) contribute to the effective Einstein-Hilbert action of gravity, see following subsection D, and \( B_4 \), as a portion of the full anomaly, is the conventional Weyl anomaly of scalar fields in \( D = 4 \).

We note that \( B_4 \) as the only dimensionless coefficient counts the number of anomalous conformal modes, in this sense, \( N(M^D) \) indeed relates to certain entropy. However, it is clearly that the single \( B_4 \) coefficient does not count the total number of modes, in fact, the total anomalous partition function \( e^{\lambda N(M^D)} \) relates to the counting function (remind the relation between the heat kernel and the counting function of the operator \( \Delta_L \)). Obviously this theory at \( 2 < d = 4 - \epsilon \) is not conformal invariant, thus as the theory flows along \( t \), the degrees of freedom are gradually coarse-grained and hence the modes-counting-number should also change with the flow and the scale, as a consequence all coefficients \( B_k \) in the series and hence the total partition function \( e^{\lambda N(M^D)} \) should count the total modes (besides the conformal modes) at certain scale \( t \), leading to the full entropy and anomaly.

Different from some classically conformal invariant theories, e.g. the string theory, in which we only need to cancel a single scale-independent \( B_k \) coefficient in order to avoid conformal anomaly. As the theory at higher than 2-dimension is not conformal invariant, the full scale-dependent anomaly \( N(M^D) \) is required to be canceled at certain scale. Fortunately, it will show in later subsection that a normalized full anomaly \( \lambda \tilde{N}(M^D) \) can converge at UV for its monotonicity, thus giving rise to a finite counter term of order \( O(\lambda) \) playing the role of a correct cosmological constant.

C. Relative Shannon Entropy and a H-Theorem for Non-Equilibrium Frame Fields

In the Ricci flow limit, i.e. the Gradient Shrinking Ricci Soliton (GSRS) configuration, the Shannon entropy \( N \) taking its maximum value \( N_\ast \), it is similar with the thermodynamics system being in a thermal equilibrium state where its entropy is also maximal. In mathematical literature of Ricci flow, it is often defined a series of relative formulas w.r.t. the extremal values taking by the flow limit GSRS or analogous thermal equilibrium state denoted by a subscript \( \ast \).

In GSRS, the covariance matrix \( \sigma_{\mu\nu} \) as 2nd central moment of the frame fields with a IR cutoff \( k \) is simply proportional to the metric

\[ \frac{1}{2} \sigma_{\mu\nu} = \frac{1}{2} \langle \delta X^\mu \delta X^\nu \rangle = \frac{1}{2\lambda} g_{\mu\nu} \int_0^{p=k} \frac{d^4p}{(2\pi)^2} \frac{1}{p^2} = -\frac{k^2}{64\pi^2} g_{\mu\nu} = \tau g_{\mu\nu}, \tag{42} \]

and then

\[ \sigma_{\mu\nu} = (\sigma_{\mu\nu}^\ast)^{-1} = \frac{1}{2\tau} g_{\mu\nu}, \tag{43} \]

which means a uniform Gaussian broadening is achieved. And in this gauge, only longitudinal part of fluctuation exists.

When the density normalized Ricci curvature is completely given by the longitudinal fluctuation \( \sigma_{\mu\nu} \), i.e. the inequality (18) saturates, giving a Gradient Shrinking Ricci Soliton (GSRS) equation

\[ R_{\mu\nu} + \nabla_\mu \nabla_\nu f = \frac{1}{2\tau} g_{\mu\nu}. \tag{44} \]

It means, on the one hand, for a general \( f(X) = \frac{1}{2} \sigma_{\mu\nu} X^\mu X^\nu \), so \( R_{\mu\nu} \) seems vanish, so the standard Ricci flow equation (10) terminates; and on the other hand, the Ricci-DeTurck flow (19) only changes the longitudinal size or volume of the manifolds but its shape keep unchanged, thus the GSRS can also be seen stop changing, up to a size or volume rescaling. Thus the GSRS is a flow limit and can be viewed as a generalized RG fixed point.

In the following, we consider relative quantities w.r.t. the GSRS configuration. Considering a general Gaussian density matrix

\[ u(X) = \det (\sigma_{\mu\nu})^{1/2} \frac{1}{(2\pi)^D/2} \exp \left( -\frac{1}{2} X^\mu \sigma_{\mu\nu} X^\nu \right), \tag{45} \]

in GSRS limit it becomes

\[ u_\ast(X) = \frac{1}{\lambda(4\pi\tau)^D/2} \exp \left( -\frac{1}{4\tau} g_{\mu\nu} X^\mu X^\nu \right). \tag{46} \]
Therefore, in GSRS, a relative density can be defined by the general Gaussian density \( u(X) \) relative to the density \( u_*(X) \) in GSRS

\[
u(X) = \frac{u}{u_*}.
\]

(47)

By using the relative density, a relative Shannon entropy \( \tilde{N} \) can be defined by

\[
\tilde{N}(M^D) = - \int d^D X u \log u - \int d^D X u \log u + \int d^D X u_* \log u_* = N - N_* = - \log Z_P \leq 0,
\]

(48)

where \( Z_P \) is nothing but the Perelman’s partition function

\[
\log Z_P = \int d^D X u \left( \frac{D}{2} - f \right) \geq 0,
\]

(49)

and \( N_* \) is the maximum Shannon entropy

\[
N_* = - \int d^D X u_* \log u_* = \int d^D X u \left( 1 + \log(4\pi \tau) \right) = \frac{D}{2\lambda} \left[ 1 + \log(4\pi \tau) \right].
\]

(50)

Since the relative Shannon entropy and the anomaly term is pure real, so the change of the partition function under diffeomorphism is non-unitary. For the coarse-graining nature of the density \( u \), it is proved that the relative Shannon entropy is monotonic non-decreasing along the Ricci flow (along \( t \))

\[
\frac{\partial \tilde{N}(M^D)}{\partial t} = - \hat{F} \geq 0,
\]

(51)

where \( \hat{F} = F - F_* \leq 0 \) is the GSRS-normalized F-functional of Perelman

\[
F = \frac{\partial N}{\partial \tau} = \int d^D X u \left( R + |\nabla f|^2 \right)
\]

(52)

with the maximum value (at GSRS limit)

\[
F_* \equiv F(u_*) = \frac{\partial N_*}{\partial \tau} = \frac{D}{2\lambda}.
\]

(53)

The inequality (51) gives an analogous H-theorem to the non-equilibrium frame fields and the irreversible Ricci flow. The entropy is non-decreasing along the Ricci flow making the flow irreversible in many aspects similar with the processes of irreversible thermodynamics, meaning that as the observation scale of the spacetime flows from short to long distance scale, the process loses information and the Shannon entropy increases. The equal sign in (51) can be taken when the spacetime configuration has flowed to a limit known as a Gradient Shrinking Ricci Soliton (GSRS), when the Shannon entropy takes its maximum value. Similarly, at the flow limit the density matrix \( u_* \) eq.(46) takes the analogous standard Maxwell-Boltzmann distribution.

D. Effective Gravity at Cosmic Scale and the Cosmological Constant

In terms of the relative Shannon entropy, the total partition function (35) of the frame fields is normalized by the GSRS extream value

\[
Z(M^D) = \frac{e^{\lambda N_* - \frac{D}{2\lambda}}}{e^{\lambda N_*}} = e^{\lambda \tilde{N} - \frac{D}{2\lambda}} = Z_P e^{\lambda N_*} = \exp \left[ \lambda \int_{M^D} d^D X u (f - D) \right].
\]

(54)

The relative Shannon entropy \( \tilde{N} \) as the anomaly vanishes at GSRS or IR scale, however, it is non-zero at ordinary lab scale up to UV where the fiducial volume of the lab is considered fixed \( \lambda \int d^4 x = 1 \). The cancellation of the anomaly at the lab scale up to UV is physically required, which leads to the counter term \( \nu(M^D_{\tau=\infty}) \) or cosmological constant. The monotonicity of \( \tilde{N} \) eq.(51) and the W-functional implies

\[
\nu(M^D_{\tau=\infty}) = \lim_{\tau \to \infty} \lambda \tilde{N}(M^D, u, \tau) = \lim_{\tau \to \infty} \lambda W(M^D, u, \tau) = \inf_{\tau} \lambda W(M^D, u, \tau) < 0,
\]

(55)
where $W$, the Perelman’s W-functional, is the Legendre transformation of $\tilde{N}$ w.r.t. $\tau^{-1}$,

$$W \equiv \tau \frac{\partial \tilde{N}}{\partial \tau} + \tilde{N} = \tau \tilde{F} + \tilde{N} = \frac{d}{d\tau} (\tau \tilde{N}).$$

(56)

In other words, the difference between the effective actions (relative Shannon entropies) at UV and IR is finite

$$\nu = \lambda (\tilde{N}_{\text{UV}} - \tilde{N}_{\text{IR}}) < 0.$$  

(57)

Perelman used his analogies: the temperature $T \sim \tau$, the (relative) internal energy $U \sim -\tau^2 \tilde{F}$, the thermodynamics entropy $S \sim -W$, and the free energy $F \sim \tau \tilde{N}$, up to proportional constant balancing the dimensions on both sides of $\sim$, the equation (56) is in analogy to the thermodynamics equation $U - TS = F$. So in this sense the W-functional is also called the W-entropy. Whether the thermodynamic analogies are real and physical, or just pure coincidences, is an important issue discussed in the next sections.

In fact $e^{\nu} < 1$ (usually called the Gaussian density [49, 50]) is a relative volume or the reduced volume $\tilde{V}(M_{\tau=\infty}^D)$ of the backwards limit manifolds introduced by Perelman, or the inverse of the initial condition of the manifolds density $u_{\tau=0}^{-1}$. A finite value of it makes an initial spacetime with unit volume from UV flow and converge to a finite $u_{\tau=0}$, and hence the manifolds finally converges to a finite relative volume/reduced volume instead of shrinking to a singular point at $\tau = 0$.

As an example, for a homogeneous and isotropic universe for which the sizes of space and time (with a “ball” radius $a_\tau$) are on an equal footing, i.e. a late epoch FRW-like metric $ds^2 = a^2_\tau (-dt_0^2 + dx_1^2 + dx_2^2 + dx_3^2)$, which is a Lorenzian shrinking soliton configuration. Note that the shrinking soliton equation $R_{\mu\nu} = \frac{D}{2\tau}g_{\mu\nu}$ it satisfies and its volume form (29) are independent to the signature, so it can be approximately given by a 4-ball value $\nu(B^4_{\infty}) \approx -0.8$ [1, 2].

So the partition function, which is anomaly canceled at UV and having a fixed-volume fiducial lab, is

$$Z(M^D) = e^{\lambda \tilde{N} - \frac{D}{2}\nu}.$$  

(58)

Since $\lim_{\tau \to 0} \tilde{N}(M^D) = 0$, so at small $\tau$, $\tilde{N}(M^D)$ can be expanded by powers of $\tau$

$$\tilde{N}(M^D) = \frac{\partial \tilde{N}}{\partial \tau} \tau + O(\tau^2) = \tau \tilde{F} + O(\tau^2)$$

$$= \int_{M^D} d^D X u R_{\mu\nu} \tau + O(\tau^2),$$

(59)

in which $\lambda \int d^D X u |\nabla f_{\tau=0}|^2 \approx \frac{D}{2}$ (at GSRS) has been used.

For $D = 4$ and small $\tau$, the effective action of $Z(M^4)$ can be given by

$$- \log Z(M^4) = S_{\text{eff}} \approx \int_{M^4} d^4 X u (2\lambda - \lambda R_0 \tau + \lambda \nu) \quad \text{(small } \tau)$$

(60)

or by taking $u = \sqrt{|g|}$ and (22), we have

$$S_{\text{eff}} = \int_{M^4} d^4 X \sqrt{|g|} \left( 2\lambda - \frac{R_0}{64\pi^2} k^2 + \lambda \nu \right) \quad \text{(small } k).$$  

(61)

The effective action can be interpreted as a low energy effective action of pure gravity. As the cutoff scale $k$ ranges from the lab scale to the solar system scale ($k > 0$), the action must recover the well-tested Einstein-Hilbert (EH) action. But at the cosmic scale ($k \to 0$), we know that the EH action deviates from observations and the cosmological constant becomes important. In this picture, as $k \to 0$, the action leaving $2\lambda + \nu$ should play the role of the standard EH action with a limit constant background scalar curvature $R_0$ plus the cosmological constant, so

$$2\lambda + \nu = \frac{R_0 - 2\lambda}{16\pi G}.$$  

(62)

While at $k \to \infty$, $\lambda \tilde{N} \to \nu$, the action leaving only the fiducial Lagrangian $\frac{D}{2}\lambda = 2\lambda$ which should be interpreted as a constant EH action without the cosmological constant

$$2\lambda = \frac{R_0}{16\pi G}.$$  

(63)
Thus we have the cosmological term
\[ \lambda \nu = -\frac{2 \Lambda}{16 \pi G} = -\rho_c. \] (64)
The action can be rewritten as an effective EH action plus a cosmological term
\[ S_{\text{eff}} = \int d^4 X \sqrt{|g|} \left( \frac{R_k}{16 \pi G} + \lambda \nu \right) \text{ (small } k), \] (65)
where
\[ \frac{R_k}{16 \pi G} = 2 \lambda - \frac{R_0}{64 \pi^2} k^2, \] (66)
which is nothing but the flow equation of the scalar curvature
\[ R_k = \frac{R_0}{1 + \frac{1}{64} \pi k^2}, \text{ or } R_\tau = \frac{R_0}{1 + \frac{2}{R_0} R_\tau}. \] (67)
Since at the cosmic scale \( k \to 0 \), the effective scalar curvature is bounded by \( R_0 \) which can be measured by the “Hubble’s constant” \( H_0 \) at the cosmic scale,
\[ R_0 = D(D-1)H_0^2 = 12 H_0^2, \] (68)
so \( \lambda \) is nothing but the critical density of the 4-spacetime Universe
\[ \lambda = \frac{3 H_0^2}{8 \pi G} = \rho_c, \] (69)
so the cosmological constant is always of order of the critical density with a “dark energy” fraction
\[ \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = -\nu \approx 0.8, \] (70)
which is not far from observations. The detail discussions about the cosmological constant problem and the observational effect in the cosmology, especially the modification of the Distance-Redshift relation leading to the acceleration parameter \( q_0 \approx -0.68 \) can be found in [1, 2, 12, 13].

If we include matter into the gravity theory, consider the entangled system in \( \mathcal{H}_\psi \otimes \mathcal{H}_X \) between the to-be-studied quantum system and the quantum reference frame fields system. 24 term in eq.(8) is normalized by the Ricci flow, by using eq.(61) and eq.(66), a matter-coupled-gravity is emerged from the Ricci flow
\[ S[\psi, X] \approx \int d^4 X \sqrt{|\det g|} \left[ \frac{1}{2} g^\mu\nu \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + 2 \lambda - \frac{R_0}{64 \pi^2} k^2 + \lambda \nu \right] = \int d^4 X \sqrt{|\det g|} \left[ \frac{1}{2} g^\mu\nu \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + \frac{R_k}{16 \pi G} + \lambda \nu \right] \] (71)

IV. THERMAL EQUILIBRIUM STATE

A. A Temporal Static Shrinking Ricci Soliton as a Thermal Equilibrium State

A Gradient Shrinking Ricci Soliton configuration as a Ricci flow limit extremizes the Shannon entropy \( N \) and the W-functional, similarly, a thermal equilibrium state also extremizes the H-functional of Boltzmann and the thermodynamic entropy, so they are naturally conjectured similar even equivalent.

When the shrinking Ricci soliton \( M^4 \) is static in the temporal part, i.e. being a product manifolds \( M^4 = M^3 \times \mathbb{R} \) and \( \delta X/\delta X_0 = 0 \), where \( X_0 \in \mathbb{R} \) is the physical time, \( \mathbf{X} = (X_1, X_2, X_3) \in M^3 \) is a 3-space gradient shrinking Ricci soliton of lower dimensions, we can prove here that the temporal static spatial part \( M^3 \) is in thermal equilibrium with the flow parameter \( \tau \) proportional to its temperature, and the manifolds density \( u \) of \( M^3 \) can be interpreted as the thermal equilibrium ensemble density.

According to Masubara’s formalism of thermal fields theory, the thermal equilibrium of the spatial frame fields can be defined by the periodicity \( \mathbf{X}(\mathbf{x},0) = \mathbf{X}(\mathbf{x},\beta) \) in their Euclidean time of the lab (remind that we start from the
Euclidean base space for the frame fields theory), where $\beta = 1/T$ is the inverse of the temperature. Now the frame fields is a mapping $\mathbb{R}^3 \times S^1 \to M^3 \times \mathbb{R}$. Then in such configuration, the $\tau$ parameter of the 3-space shrinking soliton $M^3$ becomes

$$\tau = \frac{1}{2\lambda} \int \frac{d^3p d\omega}{(2\pi)^3} \frac{1}{p^2 + \omega^2} = \frac{1}{2\lambda} T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + (2\pi n T)^2},$$

where $\omega_n = 2\pi n T$, $\int \frac{d\omega}{2\pi} = T \sum_n$ have been used. The calculation is a periodic-Euclidean-time version of (42). Since the wavefunction $\sqrt{\eta}$ of the frame fields $X_\mu$ is Gaussian, it seems almost be condensed in the ground state of the oscillator, thus $\omega_0 = 0$ dominates the Masubara sum,

$$\tau = \frac{1}{2\lambda} T \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2}. \tag{73}$$

Different from the naive notion of “temporal static” at the classical level, which means $\langle \delta X / \delta X_0 \rangle = 0$, however, the notion “temporal static” is a little subtle at the quantum level. Because there is no “absolute static” at the quantum level so the notion “temporal static” is almost frozen in the Gaussian wavepacket of $X$ can be in translational motion so $p \neq 0$, its expectation value is in general finite, i.e. $\langle \delta X(x) / \delta x_0 \rangle \propto \frac{1}{2} T < \infty$ as the equipartition energy of the translational motion. Whether or not the modes of the spatial frame fields is temporal static depends on the scale to evaluate the average of the physical clock $\langle X_0 \rangle$. The notion of “thermal static” in the sense of statistical physics is approximate at a macroscopic scale rather than a microscopic scale at which the molecules are always in motion (so does the physical clock $X_0$). The macroscopic scale of the thermal static system is such a long physical time scale $\delta \langle X_0 \rangle \gg \delta x_0$ that the averaged physical clock is almost frozen $\delta x_0 / \delta X_0 \to 0$ w.r.t. the infinitely precise lab time $x_0$, so that the thermal static condition $\langle \delta^2 X \rangle / \delta x_0 = \langle \delta X \rangle / \delta x_0 \langle \delta X \rangle / \delta x_0$ is achieved.

More precisely, we mention that the 3-space is “temporal static”, which is also in the sense of macroscopic, a IR cutoff $|p| = H_0$ being the macroscopic Hubble scale should be taken, meaning that the fluctuation modes on the 3-space should be outside the Hubble scale $0 < |p| < H_0$ and hence be frozen and temporal static, while those modes $|p| > H_0$ inside the Hubble horizon are dynamically fast vibrating, so with this spatial cutoff we have

$$\tau = \frac{1}{2\lambda} T \int_0^{p = H_0} \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} = C_3 \frac{T H_0}{2\lambda} = \frac{1}{\lambda_3} T = \frac{1}{\lambda_3 \beta}, \tag{74}$$

where

$$\lambda_3 = \frac{\lambda}{2 C_3 H_0} = \frac{\lambda}{12 \pi^2 H_0}. \tag{75}$$

Note that if we consider the temporal integral is also about a long physical time scale, the age of the universe, i.e. Hubble’s time $0 < x_0 \lesssim 1/H_0$, let $\frac{1}{12 \pi^2} \int_0^{12\pi^2 / H_0} dx_0 H_0 = 1$, so the condition $\int d^4x \lambda \equiv 1$ gives rise to

$$\int d^3x \lambda_3 = 1, \tag{76}$$

which means that $\lambda_3$ is a 3-space energy density on the static temporal slice of the lab.

So an important observation is that when $M^3$ is a shrinking Ricci soliton in a temporal static product shrinking soliton $M^3 \times \mathbb{R}$, the $\tau$ parameter of $M^3$ can be interpreted as a temperature defined by the Euclidean time periodic of the frame fields, with the coefficient of proportionality a 3-space energy density balancing the dimensions between $\tau$ and $T$. The observation gives a reason why in Perelman’s paper $\tau$ could be analogous to the temperature $T$. The same result can also be obtained if one use the Lorentzian signature for the lab/base spacetime of the frame fields theory (2). In this case the thermal equilibrium of the spatial frame fields insteadly are subject to periodicity in the imaginary Minkowskian time $X(x,0) = X(x, i\beta)$, but even though the base spacetime is Wick rotated, the path integral does not pick any imaginary $i$ factor in front of the action in (25).

### B. Thermodynamic Functions

In this subsection, we all take the temperature $T = \lambda_3 \tau$ (74), $D = 3$ and $\lambda$ replaced by $\lambda_3$. When the spatial shrinking soliton $M^3$ is in temporal static $dX_0 = 0$ and in thermal equilibrium, the partition function of the thermal
ensemble of the frame fields $X$ can be given by the trace/integration of the density matrix,

$$Z_\ast(\tau) = \lambda_3 \int d^3X u(X) = \lambda_3 \int d^3X e^{-\frac{\lambda_3}{T} \beta X} = \lambda_3 (4\pi\tau)^{3/2},$$

(77)

the normalized $u$ density can be given by

$$u_\ast(X) = \frac{1}{Z_\ast} u(X) = \frac{1}{\lambda_3 (4\pi\tau)^{3/2}} e^{-\frac{\lambda_3}{T} \beta X}.$$  

(78)

The partition function can also be consistently given by (35) with $D = 3$ in thermal equilibrium and hence the partition function of the frame fields in the shrinking soliton configuration

$$Z_\ast(\tau) = e^{\lambda_3 N_\ast(M^3)^{-\frac{3}{2}}} = \exp \left[-\lambda_3 \int d^3X u_\ast \log u_\ast - \frac{3}{2}\right] = \lambda_3 (4\pi\tau)^{3/2} = V_3 \left(\frac{4\pi\lambda_3^{1/3}}{\beta}\right)^{3/2} = Z_\ast(\beta),$$

(79)

where $V_3 = \int d^3x$ is the 3-volume with the constraint $\lambda_3 V_3 = 1$. The partition function is identified with the partition function of the canonical ensemble of ideal gas (i.e. non-interacting frame fields gas in the lab) of temperature $1/\beta$, and the interactions are effectively absorbed into the broadening of the density matrix and the correction to the gas particle mass. From the partition function, we see that the mass of the gas particle is about $\lambda_3^{1/3}$ depending on the normalization scale of the 3-volume of the lab. If the scales of the space and time are considered equally the Hubble scale, $\lambda_3^{1/3} \sim \lambda^{1/4} \sim O(10^{-3}eV)$, if we taking (75), $\lambda_3^{1/3} \sim O(MeV)$. The frame fields gas picture is qualitatively consistent with the previous consideration of the physical quantum reference frame in a lab, e.g. the electrons gas ($M_e \sim 0.5MeV$) in multi-wire proportional chamber in the lab, and photons gas ($M \sim 0$) in large scale spacetime measurements in the lab.

The physical picture of frame fields gas in thermal equilibrium lays a statistical and physical foundation to Perelman’s analogies between his functionals and thermodynamics equations as follows. The internal energy of the frame fields gas can be given similar to the standard internal energy of ideal gas $\frac{3}{2}kT$ given by the equipartition energy of translational motion. Consider $\beta$ as the Euclidean time of the flat lab, the internal energy seen from an observer in the lab is

$$E_\ast = -\frac{\partial \log Z_\ast}{\partial \beta} = \lambda_3^2 \tau^2 \frac{\partial N_\ast}{\partial \tau} = \lambda_3^2 \tau^2 F_\ast = \frac{3}{2} \lambda_3 \tau = \frac{3}{2} T,$$

(80)

in which (53) with $D = 3$ and $\lambda \rightarrow \lambda_3$ have been used.

The fluctuation of the internal energy is given by

$$\langle E_\ast^2 \rangle - \langle E_\ast \rangle^2 = \frac{\partial^2 \log Z_\ast}{\partial \beta^2} = \frac{3}{2} \lambda_3^2 \tau^2 = \frac{3}{2} T^2.$$  

(81)

The Fourier transformation of the density $u(X)$ is given by

$$\tilde{u}_\ast(K) = \int d^3X u_\ast(X) e^{-iKX} = e^{-\tau K^2},$$

(82)

since $u$ satisfies the conjugate heat equation (23), so $K^2$ is the eigenvalue of the Laplacian $-4\Delta X + R$ of the 3-space, taking the value of the F-functional,

$$K^2 = \lambda_3 \int d^3X \left(R|\Psi|^2 + 4|\nabla\Psi|^2\right) = \lambda_3 F,$$

(83)

so

$$\tilde{u}_\ast(K^2) = e^{-\lambda_3 \tau F}.$$  

(84)

For a state of energy $\lambda_3^2 T^2 F = E$, the probability density of the state can be rewritten as

$$\tilde{u}_\ast(E) = e^{-\frac{E}{2\lambda_3^2 T^2} = e^{-\frac{E}{\lambda_3 T^2}},$$

(85)

which is the standard Boltzmann’s probability distribution of the state. So we can see that the (Fourier transformed) manifolds density can be interpreted as the thermal equilibrium canonical ensemble density of the frame fields.
The free energy is given by

$$ F_* = -\frac{1}{\beta} \log Z_* = -\lambda_3 \tau \log Z_* = -\frac{3}{2} \lambda_3 \tau \log (4\pi \tau), \quad (86) $$

similar with the standard free energy of ideal gas $-\frac{3}{2} T \log T$ up to a constant.

The minus H-functional of Boltzmann at an equilibrium limit and the thermal entropy of the frame fields gas can be given by the Shannon entropy

$$ \lambda_3 N_* = S_* = -\lambda_3 \int d^3X u_* \log u_* = \frac{3}{2} [1 + \log (4\pi \tau)], \quad (87) $$

similar with the thermal entropy of fixed-volume ideal gas $\frac{3}{2} \log T + \frac{3}{2}$ up to a constant. The thermal entropy can also be consistently given by the standard formula

$$ S_* = \log Z_* - \beta \frac{\partial \log Z_*}{\partial \beta} = \frac{3}{2} [1 + \log (4\pi \tau)]. \quad (88) $$

V. APPLICATION TO THE SCHWARZSCHILD BLACK HOLE

A. The Temperature of a Schwarzschild black hole

The Schwarzschild black hole is an example of classical static shrinking Ricci soliton. A rest observer distant from it sees an approximate metric $M^3 \times \mathbb{R}$, where the spatial part $M^3$ is a shrinking Ricci soliton. The reason is as follows, because the black hole satisfies the Einstein’s equation

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \quad (89) $$

where the stress tensor is a point distributed matter with a mass $M$ at the origin $x = 0$ (seen from the infinite distant lab)

$$ T_{00} = -M \delta^{(3)}(x), \quad T_{ij} = 0 \quad (i, j = 1, 2, 3), \quad (90) $$

so we have

$$ R(x) = -8\pi G T_{\mu}^{\mu} = 8\pi G M \delta^{(3)}(x). \quad (91) $$

From the Einstein’s equation we have the Ricci curvature of $M^3$ is proportional to the metric of $M^3$

$$ R_{ij}(x) = 8\pi G T_{ij} + \frac{1}{2} g_{ij} R = \frac{1}{2} 8\pi G M \delta^{(3)}(x) g_{ij} \quad (i, j = 1, 2, 3), \quad (92) $$

the equation is nothing but a normalized shrinking Ricci soliton equation (44) for $M^3$

$$ R_{ij} = \frac{1}{2\tau} g_{ij} \quad (i, j = 1, 2, 3) \quad (93) $$

with

$$ \delta^{(3)}(x) \tau = \frac{1}{8\pi G M}, \quad (94) $$

where $\delta^{(3)}(x)$ plays the role of a 3-space density $\lambda_3$, satisfying $\int d^3x \delta^{(3)}(x) = 1$, so by using (74), we can directly read from the equation that a temperature seen by the lab’s infinite distant rest observer is

$$ T = \delta^{(3)} \tau = \frac{1}{8\pi G M}, \quad (95) $$

which is the standard Hawking’s temperature of the Schwarzschild black hole.
B. The Energy of a Schwarzschild black hole

In classical general relativity, we often mention the ADM energy of the black hole

\[ M = - \int d^3 x T_{00} = \int d^3 x M \delta^{(3)}(x), \]

which is defined w.r.t. a real time \( x_0 \) in an asymptotic flat lab. Here at the quantum level, the coordinates or frame fields are fluctuating, which gives rise to the internal energy related to the periodic of Euclidean time (\( \beta \)). And mathematically speaking, the anomaly of the trace of the stress tensor will modify the total ADM mass at the quantum level, see (36). Since the anomaly of the action of the frame fields \( \lambda_3 N_\ast \) representing the spacetime part is always real, the internal energy of the frame fields is given by the (80)

\[ E_\ast = - \frac{\partial \log Z_\ast}{\partial \beta} = \frac{3}{2} T = \frac{3}{16\pi G M}, \]

which is an extra contribution to the ADM energy. Essentially this term can be seen as a quantum correction or a part of the trace anomaly contribution to the stress tensor, thus the total energy of the black hole including the classical ADM energy and the quantum fluctuating internal energy of the metric is given by

\[ M_{BH} = - \int d^3 x \langle T_{00} \rangle = M + E_\ast = M + \frac{3}{16\pi G M}. \]

So for a macroscopic classical black hole, \( M \gg \sqrt{\frac{1}{G}} \), the first term ADM energy dominates,

\[ M_{BH} \approx M. \]

The second internal energy term is gradually unignorable for a quantum black hole. An important effect of the existence of the second term is, for a microscopic quantum black hole, it makes the total energy bound from below, the minimal energy is of order of the Planck mass

\[ M_{BH} \geq \sqrt{\frac{3}{4\pi G}} \sim O(M_p), \]

which prevents the black hole evaporating into nothing.

C. The Entropy of a Schwarzschild black hole

The entropy of the black hole comes from the uncertainty or quantum fluctuation moment of the frame fields, measured by the manifolds density \( u \) of the spacetime. For an observer in the distant lab rest frame, the static density distribute mostly in an exterior thin shell near the horizon, and sparsely in the bulk outside the horizon.

The reason is as follows. Because \( u \) density satisfies the conjugate heat equation (23) on the background of the black hole, note (91) that apart from the origin the scalar curvature \( R = 0 \), and for a classical thermal equilibrium black hole so that the temperature \( \tau \) (and hence the mass) can be seem unchanged \( \frac{\partial u}{\partial \tau} = 0 \), the conjugate heat equation becomes

\[ - \frac{\partial^2 u_k}{\partial \rho^2} + k^2 e^{2\rho} u_k = k_0^2 u_k, \]

where \( k_0 \) is the eigen-energy of the modes. By using a natural boundary condition that \( u \) vanishes at infinity, we can see that each transverse Fourier mode \( u_k \) can be considered as a free 1+1 dimensional quantum field confined in
a box, one wall of the box is at the reflecting boundary \( \rho_0 = \log \epsilon_0 \) where \( \epsilon_0 \approx 0 \), and the other wall of the box is provided by the potential
\[
V(\rho) = k^2 \epsilon^2 \rho^2,
\]
which becomes large \( V(\rho) \gg 1 \) at \( \rho > -\log k \). So we can approximate the potential by the second wall at \( \rho_w = -\log k \).

So the length of the box is given by
\[
\Delta \rho = \rho_w - \rho_0 = -\log(\epsilon_0 k).
\]

Thus the thickness of the horizon is about \( \Delta r \sim e^{\Delta \rho} \sim \epsilon_0 k \).

The density \( u_k(\rho) \) is located in the box \( \rho \in (\rho_0, \rho_w) \). In other words, the solution of \( u \) density is located mainly in a thin shell near the horizon \( \tau \in (r_H, r_H + \epsilon_0 k) \). Furthermore, the modes \( k \) is assumed normal distributed (with a tiny width related to the temperature \( \tau \)). In this picture, without solving the equation, we can approximately write down the natural solution as \( u_k(\tau) \approx \delta(|k|) \delta(\tau - r_H) \), then at finite and small \( \tau \),
\[
u_k(\tau) \approx \delta(|k|) \cdot \frac{1}{(4\pi \tau)^{1/2}} e^{-\frac{(r-H)^2}{4\tau}} = \frac{1}{(4\pi |k|^2 \tau)^{1/2}} e^{-\frac{|k|^2 \tau}{4\tau}},
\]
and by using (108) we have (up to a constant)
\[
\log u_k(\tau) \approx \frac{1}{2} \log \left(|k|^2 \tau\right).
\]

A routine calculation of the relative Shannon entropy or \( W \)-functional gives the entropy of each \( k \)-mode in the limit that the width \( \tau \) is very small,
\[
\lambda_3 \tilde{N}(u_k) = -\lambda_3 \int d^3 X u_k \log u_k
\]
\[
= \delta(|k|) \int_{r_H}^\infty 4\pi \tau^2 dr \frac{1}{(4\pi)^{1/2}} e^{-\frac{(r-H)^2}{4\tau}} \frac{1}{2} \log \left(|k|^2 \tau\right)
\]
\[
\tau \approx 0 \delta(|k|) \frac{1}{4} A \log \left(|k|^2 \tau\right),
\]
where \( A = 4\pi r_H^2 \) is the area of the horizon.

It is naturally to assume the momentum \( k \) in the horizon shell is homogeneous,
\[
|k| = |k_r| = |k_\perp|
\]
where \( k_r \) is the momentum in the radius direction and \( k_\perp \) in the transverse directions of the horizon. When we integrate over all \( k \)-modes, we have the total relative Shannon entropy weakly depending to \( \tau \)
\[
\lambda_3 \tilde{N}(u) = \lambda_3 \int d^3 k \tilde{N}(u_k)
\]
\[
= \frac{1}{4} A \int \frac{d^2 k_\perp}{(2\pi)^2} \log \left(|k_\perp|^2 \tau\right) \int dk_r \delta(k_r)
\]
\[
\approx \frac{1}{4} A \int_0^{1/\epsilon} \frac{2\pi k_\perp dk_\perp}{(2\pi)^2} \log \left(|k_\perp|^2 \tau\right)
\]
\[
= \frac{1}{4} A \times \frac{1}{2\pi} \left[ \frac{\tau}{2\epsilon^2} \left( 1 - \log \frac{\tau}{\epsilon^2} \right) \right]
\]
\[
\approx -\frac{A}{16\pi \epsilon^2},
\]
in which the transverse momentum is effectively cut off at an inverse of a fundamental UV length scale \( \epsilon^2 \).

The relative Shannon entropy gives an area law of the black hole entropy. To determine the UV length cutoff \( \epsilon^2 \), we need to consider the scale at which the relative entropy is defined to be zero (not only the black hole is locally thermal equilibrium, but also the asymptotic background spacetime is globally thermal equilibrium), thus we need to consider the flow of the asymptotic background spacetime. A natural choice of a thermal equilibrium Ricci flow limit of the background spacetime is an asymptotic homogeneous and isotropic Hubble universe with scalar curvature...
\( R_0 = D(D-1)H_0^2 = 12H_0^2 \) at scale \( t_{UV} \) where we could consider and normalize the relative entropy to be zero, since there is no information of the local shape distortions in such background because of the vanishing of its Weyl curvature, while the global Riemannian curvature is non-zero which conveys the information of its global volume shrinking. Under the definition, taking the normalized Shrinking Ricci soliton equation (44) and (22), we have

\[
\tau_{UV} = -t_{UV} = \frac{D}{2H_0} = \frac{1}{64\pi^2\lambda} k_{UV}^2, \tag{110}
\]

by using the critical density (69), which gives a natural cutoff corresponding to the scale \( t_{UV} \),

\[
e^2 = k_{UV}^{-2} = \frac{1}{D\pi} G = \frac{1}{4\pi} G. \tag{111}
\]

This is exactly the Planck scale, which is a natural cutoff scale induced from the Hubble scale \( H_0 \) and \( \lambda \) of the framework. However, it is worth stressing that the Planck scale is not the absolute fundamental scale of the theory, it only has meaning w.r.t. the asymptotic Hubble scale. The only fundamental scale of the theory is the critical density \( \lambda \) which is given by a combination of both the Planck scale and Hubble scale, but each individual Planck or Hubble scale does not have absolute meaning. The UV (Planck) cutoff scale could tend to infinity while the complementary (Hubble) scale correspondingly tends to zero (asymptotic flat background), keeping \( \lambda \) finite and fixed.

At this point, if we define a zero-relative-entropy for an asymptotic Hubble universe of scalar curvature \( R_0 \), then the black hole in this asymptotic background has a non-zero thermodynamic entropy

\[
S = -\lambda_3 \tilde{N}(u) = \frac{A}{4G}, \tag{112}
\]

besides the bulk GSRS background entropy \( \lambda_3 N_* = S_* \), eq.(88), of the bulk spacetime background. Combine the relative Shannon entropy \( \tilde{N} \) and the bulk thermal entropy \( N_* \), by using the total partition function eq.(35), \( Z(M^2) = e^{\lambda_3 N - \frac{c}{2}} \), we reproduce (98)

\[
M_{BH} = -\frac{\partial \log Z}{\partial \beta} = M + \frac{3}{2} T, \tag{113}
\]

in which eq.(48) and \( A = 4\pi r_H^2 = 16\pi G^2 M^2 = \frac{g^2}{4\pi} \) have been used.

**VI. CONCLUSIONS**

In this paper, we have proposed a statistical fields theory underlying Perelman’s seminal analogies between his geometric functionals and the thermodynamic functions. The theory is based on a \( d = 4 \) quantum non-linear sigma model, interpreted as a quantum reference frame. When we quantize the theory at the Gaussian approximation, the wavefunction \( \Psi(X) \) and hence the density matrix \( u(X) = \Psi^*(X)\Psi(X) \) (13) can be written down explicitly. Based on the density matrix, the Ricci flow of the frame fields (10) and the generalized Ricci-DeTurck flow (19) of the frame fields endowed with the density matrix is discussed. And further more, we find that the density matrix has profound statistical and geometric meanings, by using it, the Riemannian spacetime \( (M^D, g) \) as the target space of NLSM is generalized to a density Riemannian spacetime \( (M^D, g, u) \). The density matrix \( u(X, \tau) \), satisfying a conjugate heat equation (23), not only describes a (coarse-grained) probability density of finding frame fields in a local volume, but also describes a volume comparison between a local volume and the fiducial one.

For the non-isometric nature of the Ricci or Ricci-DeTurck flow, the classical diffeomorphism is broken down at the quantum level. By the functional integral quantization method, the change of the measure of the functional integral can be given by using a Shannon entropy \( N \) in terms of the density matrix \( u(X, \tau) \). The induced trace anomaly and its relation to the anomalies in conventional gravity theories are also discussed. As the Shannon entropy flows monotonically to its maximal value \( N_* \) in a limit called Gradient Shrinking Ricci Soliton (GSRS), a relative density \( u_\tau \) and relative Shannon entropy \( \tilde{N} = N - N_* \) can be defined w.r.t. the flow limit. The relative Shannon entropy gives a statistical interpretation underlying Perelman’s partition function (48). And the monotonicity of \( \tilde{N} \) along the Ricci flow gives an analogous H-theorem (51) for the frame fields system. As a side effect, the meanings on the gravitational side of the theory is also discussed, in which a cosmological constant \( \lambda \nu B_\infty^4 \approx -0.8 \rho_c \) as a UV counter term of the anomaly must be introduced.

We find that a temporal static GSRS, \( M^2 \) as a space-like subspace of the GSRS spacetime \( M^4 = M^2 \times \mathbb{R} \) is in a thermal equilibrium state, in which the temperature is proportional to the \( \tau \) parameter of \( M^3 \) (74). Base on the
statistical interpretation of the density matrix \( u(X, \tau) \), we find that the thermodynamic partition function (77) at the Gaussian approximation is just a partition function of ideal gas of the frame fields with mass \( \lambda_{3/2}^{1/3} \). In this physical picture of canonical ensemble of frame fields gas, several thermodynamic functions, including the internal energy (80), the free energy (86), the thermodynamic entropy (87), and the ensemble density (85) etc. can be calculated explicitly agreeing with the ideal gas picture, which gives an underlying statistical foundation to Perelman’s analogous formulas.

We find that the statistical fields theory of quantum reference frame can be used to give a possible underlying microscopic origin of the spacetime thermodynamics. The standard results of the thermodynamics of the Schwarzschild black hole, including the black hole temperature and area law of black hole entropy can be successfully reproduced. And we find that when the fluctuation internal energy of the metric is taken into account in the total ADM energy, the energy of the black hole has a lower bound of order of the Planck energy, which avoid the quantum black hole evaporating into nothing.

The paper is an attempt to discuss the profound relations between these three fundamental notions: the diffeomorphism anomaly, spacetime thermodynamics and the gravity. In the spirit of classical general relativity, if we trust the Equivalence Principle, one can not in principle figure out whether one is in an absolute accelerating frame or in an absolute gravitational background, which leads to a general covariance principle or diffeomorphism invariance of the gravitational theory. However, at the quantum level, the issue is a little subtle. If an observer in an accelerating frame sees the Unruh effect, i.e. thermal particles are creating in the vacuum, which seems leading to the unitarily inequivalence between the vacuums of e.g. an inertial frame and accelerating frame, and hence the diffeomorphism invariance is seen breakdown discussed as the anomaly in the paper. The treatment of the anomaly in the paper is that, the anomaly is only cancelled in an observer’s lab up to UV scale, where the frame can be considered classical, rigid and cold, while at general scale the anomaly is not completely cancelled. Whether one can figure out that he/she is in an absolute accelerating frame by observing the existence of the anomaly term at general scale? (e.g. by observing the vacuum thermal particle creation and hence the non-unitarity). We argue that if the answer is still no in the spirit of general relativity, the anomaly term must be also equivalently interpreted as the effects of spacetime thermodynamics and gravity. Because the anomaly in terms of the density matrix is depicted by the 2nd order moment fluctuation \( \sigma_{\mu\nu} \) of the coordinates, which is physically equivalent to the curvature of the spacetime (see (18)) and thermal broadening (see (74)). In this sense, the validity of the classical Equivalence Principle would be generalized to the quantum level, more precisely, if one observes the 2nd order moment of the spacetime (e.g. non-trivial density matrix or other Unruh-like phenomenon), one in principle still can not figure out whether one is in an absolute curved spacetime or in an absolute accelerating frame or just in an absolute thermal spacetime, they have no absolute physical meanings any more.

Data availability statement

All data that support the findings of this study are included within the article.

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