Properties of fuzzy set spaces with $L_p$ metrics

Huan Huang

Department of Mathematics, Jimei University, Xiamen 361021, China

Abstract

In this paper, we discuss the properties of the spaces of fuzzy sets in a metric space with $L_p$-type $d_p$ metrics, $p \geq 1$. Firstly, we give the characterizations of compactness in fuzzy set space with $d_p$ metrics. Then we present the completions of fuzzy set spaces with $d_p$ metrics. The $d_p$ metrics are well-defined if and only if a function induced by the Hausdorff metric is measurable. In this paper, we give some fundamental conclusions on the measurability of this function.

Keywords: $L_p$ metric; Hausdorff metric; Compactness; Completion; Measurability

1. Introduction

The $L_p$-type metrics are widely used in theoretical research and practical applications. The class of $d_p$ metrics are a kind of $L_p$-type metrics. The $d_p$ metrics are commonly used metrics on fuzzy sets [1, 2, 18, 21].

Compactness is a fundamental property in both theory and applications [6, 14, 17]. The characterizations of compactness for fuzzy set space with $d_p$ metrics have attracted attentions of scholars [15, 22, 23]. The completion of a given metric space is an important topic in analysis.

In [7], we give the characterizations of total boundedness, relative compactness and compactness for fuzzy set space with $d_p$ metrics. We also give the completions of various fuzzy set spaces with $d_p$ metrics. All the fuzzy
sets involved in the conclusions in [7] are fuzzy sets in the $m$-dimensional Euclidean space $\mathbb{R}^m$.

It is natural to consider spaces of fuzzy sets in a metric space [4, 5, 13]. In this paper, we discuss the properties spaces of fuzzy sets in a general metric space with $d_p$ metrics.

We discuss the properties of the $d^*_p$ metric ($d_p$ metrics) and the $H_{end}$ metric including the relationship between them. Based on these results and the characterizations of total boundedness, relative compactness and compactness in fuzzy set space with $d_p$ metrics given in [9], We present the characterizations of total boundedness, relative compactness and compactness for space of fuzzy sets in a metric space with $d_p$ metrics. These results reveal a connection between a set being compact in the sense of the $d_p$ metrics and compact in the sense of the endograph metrics. The results on the characterizations of total boundedness, relative compactness and compactness in this paper generalize the corresponding results in [7]. Furthermore, using the results in this paper, we give new characterizations of total boundedness, relative compactness and compactness for space of fuzzy sets in $\mathbb{R}^m$.

We construct completions of fuzzy set spaces in a metric space with $d_p$ metrics. These conclusions on the completions of the spaces of fuzzy set in a metric space $(X,d)$ apply to not only the cases that $X$ is a complete metric space but also the cases that $X$ is an incomplete metric space. The fuzzy sets involved in the corresponding results in [7] is fuzzy sets in $\mathbb{R}^m$, which is a complete space. The results on completions of fuzzy set spaces in this paper improve the corresponding results in [7].

The $d_p$ metrics are well-defined if and only if a function induced by the Hausdorff metric is measurable. So it is important to discuss the measurability of this function. In [8], we give some conclusions on this topic. In this paper, we give proofs to the positive conclusions and counterexamples to the negative conclusion. Further, we make great improvements to these conclusions. The conclusions on measurability of the function in this paper pointed out the cases in which the $d_p$ metrics are well-defined, and, therefore, indicate the properties of the $L_p$-type metrics can be used in these cases. So these conclusions are fundamental for relevant studies of the $d_p$ metrics.

The remainder of this paper is organized as follows. In Section 2, we recall and give some basic notions and results related to fuzzy sets and metrics on them. In Section 3, we discuss the properties and relation of the $d^*_p$ metric and the $H_{end}$ metric. In Section 4, we give the characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X)\rho, d_p)$. In
2. Fuzzy sets and metrics on them

In this section, we recall and give some notions and results related to fuzzy sets and metrics on them. Readers can refer to [1, 2, 16, 19–21] for studies and applications of fuzzy sets.

Let \( \mathbb{N} \) be the set of all natural numbers, and let \( \mathbb{R}^m \) be the \( m \)-dimensional Euclidean space.

In this paper, if not specifically mentioned, we suppose that \( X \) is a metric space endowed with a metric \( d \).

Let \( K(X) \) and \( C(X) \) denote the set of all non-empty compact subsets of \( X \) and the set of all non-empty closed subsets of \( X \), respectively.

Let \( F(X) \) denote the set of all fuzzy sets in \( X \). A fuzzy set \( u \in F(X) \) can be seen as a function \( u : X \to [0, 1] \). A subset \( S \) of \( X \) can be seen as a fuzzy set in \( X \). If there is no confusion, the fuzzy set in \( X \) corresponding to \( S \) is often denoted by \( \chi_S \); that is,

\[
\chi_S(x) = \begin{cases} 
1, & x \in S, \\
0, & x \in X \setminus S.
\end{cases}
\]

For simplicity, for \( x \in X \), we will use \( \hat{x} \) to denote the fuzzy set \( \chi_{\{x\}} \) in \( X \).

In this paper, if we want to emphasize a specific metric space \( X \), we will write the fuzzy set in \( X \) corresponding to \( S \) as \( S_{F(X)} \), and the fuzzy set in \( X \) corresponding to \( \{x\} \) as \( \hat{x}_{F(X)} \).

For \( u \in F(X) \), let \( [u]_\alpha \) denote the \( \alpha \)-cut of \( u \), i.e.

\[
[u]_\alpha = \begin{cases} 
\{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\
\text{supp } u = \{u > 0\}, & \alpha = 0,
\end{cases}
\]

where \( \text{supp } u \) denotes the topological closure of \( S \) in \( (X, d) \).

For \( u \in F(X) \), define

\[
\text{end } u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\},
\]
send $u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\} \cap ([u]_0 \times [0, 1])$. 

end $u$ and send $u$ are called the endograph of $u$ and the sendograph of $u$, respectively.

Let $F_{USC}^1(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0, 1]$, i.e.,

$$F_{USC}^1(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}.$$

We introduce some subclasses of $F_{USC}^1(X)$, which will be discussed in this paper. Define

$$F_{USCB}^1(X) := \{u \in F_{USC}^1(X) : [u]_0 \in K(X)\},$$
$$F_{USCG}^1(X) := \{u \in F_{USC}^1(X) : [u]_\alpha \in K(X) \text{ for all } \alpha \in (0, 1]\}.$$

Let $(X, d)$ be a metric space. We use $H$ to denote the Hausdorff distance on $C(X)$ induced by $d$, i.e.,

$$H(U, V) = \max\{H^*(U, V), H^*(V, U)\}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U, V) = \sup_{u \in U} d(u, V) = \sup_{u \in U} \inf_{v \in V} d(u, v).$$

The metric $\overline{d}$ on $X \times [0, 1]$ is defined as

$$\overline{d}((x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.$$ 

If there is no confusion, we also use $H$ to denote the Hausdorff distance on $C(X \times [0, 1])$ induced by $\overline{d}$.

**Remark 2.1.** $\rho$ is said to be a metric on $Y$ if $\rho$ is a function from $Y \times Y$ into $\mathbb{R}$ satisfying positivity, symmetry and triangle inequality. At this time, $(Y, \rho)$ is said to be a metric space.

$\rho$ is said to be an extended metric on $Y$ if $\rho$ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, $(Y, \rho)$ is said to be an extended metric space.

We can see that for arbitrary metric space $(X, d)$, the Hausdorff distance $H$ on $K(X)$ induced by $d$ is a metric. So the Hausdorff distance $H$ on
$K(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$ is a metric. In these cases, we call the Hausdorff distance the Hausdorff metric.

The Hausdorff distance $H$ on $C(X)$ induced by $d$ on $X$ is an extended metric, but probably not a metric, because $H(A,B)$ could be equal to $+\infty$ for certain metric space $X$ and $A, B \in C(X)$. Clearly, if $H$ on $C(X)$ induced by $d$ is not a metric, then $H$ on $C(X \times [0,1])$ induced by $\overline{d}$ is also not a metric. So the Hausdorff distance $H$ on $C(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$ is an extended metric but probably not a metric. In the cases that the Hausdorff distance $H$ is an extended metric, we call the Hausdorff distance the Hausdorff extended metric.

We can see that $H$ on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and then the same is $H$ on $C(\mathbb{R}^m \times [0,1])$.

In this paper, for simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the **Hausdorff metric**.

The $d_\infty$ metric on $F^1_{USC}(X)$ is defined as

$$d_\infty(u, v) := \sup\{H([u]_\alpha, [v]_\alpha) : \alpha \in [0,1]\}.$$  

The endograph metric $H_{\text{end}}$ and the sendograph metric $H_{\text{send}}$ can be defined on $F^1_{USC}(X)$ as usual. For $u, v \in F^1_{USC}(X),$

$$H_{\text{end}}(u, v) := H(\text{end } u, \text{end } v),$$

$$H_{\text{send}}(u, v) := H(\text{send } u, \text{send } v).$$

The endograph metric $H_{\text{end}}$ and the sendograph metric $H_{\text{send}}$ are defined by using the Hausdorff metric on $C(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$.

Clearly for $u, v \in F^1_{USC}(X),$

$$d_\infty(u, v) \geq H_{\text{send}}(u, v) \geq H_{\text{end}}(u, v).$$

**Remark 2.2.** We can see that $H_{\text{end}}$ is a metric on $F^1_{USC}(X)$ with $H_{\text{end}}(u, v) \leq 1$ for all $u, v \in F^1_{USC}(X)$. Both $d_\infty$ and $H_{\text{send}}$ are metrics on $F^1_{USC}(X)$. However, each one of $d_\infty$ and $H_{\text{send}}$ on $F^1_{USC}(X)$ is an extended metric but probably not a metric. See also Remark 3.3 in [8] (We made a misprint in the last sentence of Remark 3.3 in [8]. The “$H_{\text{end}}$” must be deleted from this sentence).

We can see that both $d_\infty$ and $H_{\text{send}}$ on $F^1_{USCG}(\mathbb{R}^m)$ are not metrics, they are extended metrics.
For simplicity, in this paper, we call $H_{\text{send}}$ on $F_{USC}^1(X)$ the $H_{\text{send}}$ metric or the sendograph metric $H_{\text{send}}$. We call $d_\infty$ on $F_{USC}^1(X)$ the $d_\infty$ metric or the supremum metric $d_\infty$.

For $u, v \in F_{USC}^1(X)$, the $d_p$ distance given by

$$d_p(u, v) = \left( \int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p}$$

is well-defined if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0, 1]$. In the sequel, we suppose that the $d_p$ distance is with $p \geq 1$.

Since $H([u]_\alpha, [v]_\alpha)$ could be a non-measurable function of $\alpha$ on $[0, 1]$ (see Example 2.13 in [12]), we introduce the $d_p^*$ distance on $F_{USC}^1(X)$, $p \geq 1$, in [8], which is defined by

$$d_p^*(u, v) := \inf \left\{ \left( \int_0^1 f(\alpha)^p \, d\alpha \right)^{1/p} : f \text{ is a measurable function from } [0, 1] \text{ to } \mathbb{R} \cup \{+\infty\}; \right.$$  

$$f(\alpha) \geq H([u]_\alpha, [v]_\alpha) \text{ for } \alpha \in [0, 1] \}$$

for $u, v \in F_{USC}^1(X)$.

We can see that if $d_p(u, v)$ is well-defined for $u, v \in F_{USC}^1(X)$, then $d_p^*(u, v) = d_p(u, v)$. The $d_p^*$ distance is an expansion of the $d_p$ distance on $F_{USC}^1(X)$.

**Remark 2.3.** In Section 7, we give some fundamental conclusions for the measurability of the function $H([u]_\alpha, [v]_\alpha)$ of $\alpha$ on $[0, 1]$.

It is shown that the function $H([u]_\alpha, [v]_\alpha)$ of $\alpha$ on $[0, 1]$ is measurable when $u, v \in F_{USC}^1(\mathbb{R}^m)$ (see Proposition 7.2). So the $d_p$ distance is well-defined on $F_{USC}^1(\mathbb{R}^m)$.

It is shown that the function $H([u]_\alpha, [v]_\alpha)$ of $\alpha$ on $[0, 1]$ is measurable when $u, v \in F_{USCG}^1(X)$ (see Proposition 7.7). So the $d_p$ distance is well-defined on $F_{USCG}^1(X)$.

In Section 7, we improve Proposition 7.2 (See Corollary 7.19), and improve Proposition 7.7 (see Theorem 7.15 and Corollary 7.20). In the sequel, we write $d_p(u, v)$ rather than $d_p^*(u, v)$ when $d_p(u, v)$ is well-defined.

In [8], we point out that $d_p^*$ on $F_{USC}^1(X)$ is an extended metric but probably not a metric (See Theorem 3.1 and Remark 3.3 in [8]).
The $d_p$ distance on $F_{USCG}^1(X)$ is an extended metric but probably not a metric because $d_p(u, v) = +\infty$ could happen for $u, v \in F_{USCG}^1(X)$. For example, define $u \in F(\mathbb{R})$ by

$$u(x) = \begin{cases} 
0, & x < 1, \\
1, & x = 1, \\
1/n, & x \in (n^2, (n + 1)^2], n = 1, 2, \ldots
\end{cases}$$

Then $u \in F_{USCG}^1(\mathbb{R})$, $\widehat{1}_{F(\mathbb{R})} \in F_{USCB}^1(\mathbb{R}) \subseteq F_{USCG}^1(\mathbb{R})$, and $d_p(u, \widehat{1}_{F(\mathbb{R})}) = +\infty$.

We can see that the $d_p$ distance on $F_{USCB}^1(X)$ is a metric; the $d_p$ distance on $F_{USCG}^1(\mathbb{R}^m)$ is an extended metric but not a metric; the $d_p$ distance on $F_{USC}^1(\mathbb{R})$ is an extended metric but not a metric.

**Remark 2.4.** In this paper, for simplicity, we refer to both the $d_p^*$ extended metric and the $d_p$ metric as the $d_p$ metric, and both the $d_p^*$ extended metric and the $d_p$ metric as the $d_p$ metric.

We introduce the following subset of $F_{USC}^1(X)$

- $F_{USCG}^1(X)^p := \{ u \in F_{USCG}^1(X) : d_p(u, \widehat{x}_0) = (\int_0^1 H([u]_\alpha, \{x_0\})^p d\alpha)^{1/p} < +\infty \}$, where $x_0$ is a point in $X$.

The definition of $F_{USCG}^1(X)^p$ does not depend on the choice of $x_0$. We can see that the $d_p$ distance is a metric on $F_{USCG}^1(X)^p$.

Clearly,

$$F_{USCB}^1(X) \subseteq F_{USCG}^1(X)^p \subseteq F_{USCG}^1(X) \subseteq F_{USC}^1(X).$$

We use $(\tilde{X}, \tilde{d})$ to denote the completion of $(X, d)$. We see $(X, d)$ as a subspace of $(\tilde{X}, \tilde{d})$. Let $S \subseteq \tilde{X}$. The symbol $\tilde{S}$ is used to denote the closure of $S$ in $(\tilde{X}, \tilde{d})$.

As defined previously, we have $K(\tilde{X}), C(\tilde{X}), F_{USC}^1(\tilde{X}), F_{USCG}^1(\tilde{X})$, etc. according to $(\tilde{X}, \tilde{d})$. For example,

$$F_{USC}^1(\tilde{X}) := \{ u \in F(\tilde{X}) : [u]_\alpha \in C(\tilde{X}) \text{ for all } \alpha \in [0, 1] \},$$
$$F_{USCG}^1(\tilde{X}) := \{ u \in F(\tilde{X}) : [u]_\alpha \in K(\tilde{X}) \text{ for all } \alpha \in (0, 1] \}.$$

If there is no confusion, we also use $H$ to denote the Hausdorff metric on $C(\tilde{X})$ induced by $\tilde{d}$. We also use $H$ to denote the Hausdorff metric on
\(C(\bar{X} \times [0,1])\) induced by \(\tilde{d}\). We also use \(H_{\text{end}}\) to denote the endograph metric on \(F^1_{\text{USC}}(\bar{X})\) given by using \(H\) on \(C(\bar{X} \times [0,1])\). We also use \(d_p\) to denote the \(d_p\) metric on \(F^1_{\text{USCG}}(\bar{X})\).

Define \(f : F^1_{\text{USCG}}(X) \to F^1_{\text{USCG}}(\bar{X})\) as follows: for \(u \in F^1_{\text{USCG}}(X)\),

\[
f(u)(t) = \begin{cases} u(t), & t \in X, \\ 0, & t \in \bar{X} \setminus X. \end{cases}
\]

Then \([f(u)]_\alpha = [u]_\alpha\) for all \(\alpha \in (0,1]\), and so \(f(u) \in F^1_{\text{USCG}}(\bar{X})\). We can see that for \(\rho = d_\infty, d_p, H_{\text{end}}, H_{\text{send}}, \rho(u,v) = \rho(f(u), f(v))\). So for \(\rho = d_\infty, d_p, H_{\text{send}}, (F^1_{\text{USCG}}(X), \rho)\) can be embedded as an extended metric subspace of \((F^1_{\text{USCG}}(\bar{X}), \rho)\). \((F^1_{\text{USCG}}(X), H_{\text{end}})\) can be embedded as a metric subspace of \((F^1_{\text{USCG}}(\bar{X}), H_{\text{end}})\).

In this paper, we see \((F^1_{\text{USCG}}(X), H_{\text{end}})\) as a metric subspace of \((F^1_{\text{USCG}}(\bar{X}), H_{\text{end}})\). We see \((F^1_{\text{USCG}}(X)^p, d_p)\) as a metric subspace of \((F^1_{\text{USCG}}(\bar{X})^p, d_p)\). We see \((F^1_{\text{USCG}}(X), d_p)\) as an extended metric subspace of \((F^1_{\text{USCG}}(\bar{X}), d_p)\).

### 3. Properties of \(d_p^*\) metric and \(H_{\text{end}}\) metric

In this section, we give the relationship between the \(d_p^*\) metric and the \(H_{\text{end}}\) metric. We illustrate the relations between the property that \(U\) is uniformly \(p\)-mean bounded and other properties of \(U\). We list the properties of \((F^1_{\text{USCG}}(X), H_{\text{end}})\) obtained in [12] which is useful in this paper.

**Theorem 3.1.** Let \((X,d)\) be a metric space.

(i) For \(u,v \in F^1_{\text{USC}}(X)\),

\[
d_p^*(u,v) \geq \left(\frac{H_{\text{end}}(u,v)^{p+1}}{p+1}\right)^{1/p}.
\]

(ii) For \(u \in F^1_{\text{USC}}(X)\) and a sequence \(\{u_n\}\) in \(F^1_{\text{USC}}(X)\), if \(d_p^*(u_n,u) \to 0\), then \(H_{\text{end}}(u_n,u) \to 0\).

**Proof.** To show (i), we only need to show that for each \(r > 0\), if \(H_{\text{end}}(u,v) > r\) then \(d_p^*(u,v) \geq \left(\frac{r^{p+1}}{p+1}\right)^{1/p}\).

Let \(r > 0\). Assume that \(H_{\text{end}}(u,v) > r\). Then without loss of generality we suppose that \(H^*(\text{end } u, \text{end } v) > r\), then there is an \((x, \beta) \in \text{end } u\), such
that \( d((x, \beta), \text{end} v) > r \). This implies that \( \beta > r \) and \( d(x, [v]_\alpha) > r - (\beta - \alpha) \)
when \( \alpha \in [\beta - r, \beta] \). Hence \( H^*([u]_\alpha, [v]_\alpha) > r - (\beta - \alpha) \)
when \( \alpha \in [\beta - r, \beta] \).

Let \( f \) be a measurable function on \([0, 1]\) with \( f(\alpha) \geq H([u]_\alpha, [v]_\alpha) \) for \( \alpha \in [0, 1] \). Then

\[
\left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} \geq \left(\int_{\beta-r}^\beta f(\alpha)^p \, d\alpha\right)^{1/p} \\
> \left(\int_{\beta-r}^\beta (r - (\beta - \alpha))^p \, d\alpha\right)^{1/p} \\
= \left(\frac{r^{p+1}}{p+1}\right)^{1/p}.
\]

So \( d_p^*(u, v) \geq \left(\frac{r^{p+1}}{p+1}\right)^{1/p} \).

(ii) follows immediately from (i).

\[\square\]

**Remark 3.2.** Clearly, if \( d_p(u, v) \) is well-defined (at this time \( d_p^*(u, v) = d_p(u, v) \)), then \( d_p^*(u, v) \) can be replaced by \( d_p(u, v) \) in Theorem 3.1.

The “=” can be obtained in (2).

**Example 3.3.** Define \( u \) and \( v \) in \( F_{USCB}^1(\mathbb{R}) \) as

\[
u(x) = \begin{cases} 
1, & x = 0, \\
0.5 - x, & x \in (0, 0.5], \\
0, & \text{otherwise},
\end{cases}
\]

\[v(x) = \begin{cases} 
1, & x = 0, \\
0.5, & x \in (0, 0.5], \\
0, & \text{otherwise}.
\end{cases}
\]

Then \( H_{\text{end}}(u, v) = 0.5 \) and

\[H([u]_\alpha, [v]_\alpha) = \begin{cases} 
0, & \alpha \in (0.5, 1], \\
\alpha, & \alpha \in [0, 0.5].
\end{cases}
\]

Thus \( d_p(u, v) = (\int_0^{0.5} \alpha^p \, d\alpha)^{1/p} = \left(\frac{0.5^{p+1}}{p+1}\right)^{1/p} = \left(\frac{H_{\text{end}}(u,v)^{p+1}}{p+1}\right)^{1/p}. \)

The following two concepts are essentially proposed by Diamond and Kloeden [1] and Ma [15], respectively.
Definition 3.4. [1] Let \( u \in F_{\text{USCG}}^{1}(X)^{p} \). If for given \( \varepsilon > 0 \), there is a \( \delta(u, \varepsilon) \in (0, 1] \) such that for all \( 0 \leq h < \delta \)

\[
\left( \int_{h}^{1} H([u]_{\alpha}, [u]_{\alpha-h})^{p} \, d\alpha \right)^{1/p} < \varepsilon,
\]

where \( 1 \leq p < +\infty \), then we say \( u \) is \( p \)-mean left-continuous.

Suppose that \( U \) is a nonempty set in \( F_{\text{USCG}}^{1}(X)^{p} \). If the above inequality holds uniformly for all \( u \in U \), then we say \( U \) is \( p \)-mean equi-left-continuous.

Definition 3.5. [15] A set \( U \) in \( F_{\text{USCG}}^{1}(X)^{p} \) is said to be uniformly \( p \)-mean bounded if there exist a constant \( M > 0 \) and an \( x_{0} \in X \) such that \( d_{p}(u, x_{0}) \leq M \) for all \( u \in U \).

We say that a set \( U \) is bounded in a metric space \((Y, \rho)\) if and only if there is an \( M > 0 \) such that \( \sup \{ \rho(x, y) : x, y \in U \} \leq M \).

Clearly, \( U \) in \( F_{\text{USCG}}^{1}(X)^{p} \) is uniformly \( p \)-mean bounded is equivalent to \( U \) is bounded in \((F_{\text{USCG}}^{1}(X)^{p}, d_{p})\).

Let \( U \) be a set in \( F_{\text{USCG}}^{1}(X)^{p} \). The following Lemma 3.6 and Theorem 3.7 illustrate the relations between the property that \( U \) is uniformly \( p \)-mean bounded and other properties of \( U \).

Lemma 3.6. Let \( U \) be a subset of \( F_{\text{USCG}}^{1}(X)^{p} \). If \( U \) is uniformly \( p \)-mean bounded, then for each \( h \in (0, 1] \), \( U(h) \) is bounded in \((X, d)\).

Proof. We proceed by contradiction. Assume that there is an \( h_{0} \in (0, 1] \) such that \( U(h_{0}) \) is not bounded in \((X, d)\). This means that \( \sup \{ H([u]_{h_{0}}, \{x_{0}\}) : u \in U \} = +\infty \). Note that for each \( u \in U \),

\[
\left( \int_{0}^{1} H([u]_{\alpha}, \{x_{0}\})^{p} \, d\alpha \right)^{1/p} \geq \left( \int_{0}^{h_{0}} H([u]_{\alpha}, \{x_{0}\})^{p} \, d\alpha \right)^{1/p} \geq h_{0} \cdot H([u]_{h_{0}}, \{x_{0}\}).
\]

Thus \( \sup \left\{ \left( \int_{0}^{1} H([u]_{\alpha}, \{x_{0}\})^{p} \, d\alpha \right)^{1/p} : u \in U \right\} = +\infty \), which contradicts the assumption that \( U \) is uniformly \( p \)-mean bounded.

Theorem 3.7. Let \( U \) be a subset of \( F_{\text{USCG}}^{1}(X)^{p} \). If \( U \) is \( p \)-mean equi-left-continuous, then the following three properties are equivalent:

(i) There exists an \( h \in (0, 1] \) such that \( U(h) \) is bounded in \((X, d)\);
(ii) For each \( h \in (0, 1] \), \( U(h) \) is bounded in \((X, d)\);
(iii) \( U \) is uniformly \( p \)-mean bounded.
Proof. (i)⇒(iii). Assume that (i) is true, i.e. there is an \( h_1 \in (0, 1) \) such that \( U(h_1) \) is bounded in \((X, d)\). Then there exists an \( L > 0 \) such that \( \sup \{ d(x, x_0) : x, y \in U(h_1) \} \leq L \). Put \( M = L \cdot (1 - h_1)^{1/p} \). Then for all \( h \in [h_1, 1] \) and \( u \in U \),

\[
\left( \int_h^1 H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} \leq L \cdot (1 - h_1)^{1/p} = M. \tag{3}
\]

Since \( U \) is \( p \)-mean equi-left-continuous, there is an \( h_2 > 0 \) such that for all \( h \in [0, h_2] \) and \( u \in U \),

\[
\left( \int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} < 1 \tag{4}
\]

Choose an \( h \leq \min\{1 - h_1, h_2\} \) satisfying \( 1/h = N \in \mathbb{N} \). Then by (4) for \( k = 1, \ldots, N - 1 \) and \( u \in U \),

\[
\left| \left( \int_{kh}^{(k+1)h} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} - \left( \int_{(k-1)h}^{kh} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} \right| \\
= \left| \left( \int_{kh}^{(k+1)h} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} - \left( \int_{kh}^{(k+1)h} H([u]_{\alpha-h}, \{x_0\})^p \, d\alpha \right)^{1/p} \right| \\
\leq \left( \int_{kh}^{(k+1)h} H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \leq \left( \int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} < 1,
\tag{5}
\]

and thus by (3) and (5), for all \( u \in U \),

\[
\left( \int_0^1 H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} \leq \sum_{k=0}^{N-1} \left( \int_{kh}^{(k+1)h} H([u]_\alpha, \{x_0\})^p \, d\alpha \right)^{1/p} \\
< M + \cdots + (M + (N - 1)) \\
= N \cdot M + N(N - 1)/2,
\]

and so (iii) is true.
(iii)⇒(ii) follows from Lemma 3.6.
(ii)⇒(i) is obvious.

Remark 3.8. From Corollary 5.1 and Theorem 5.5, we know that for a $p$-mean equi-left-continuous set $U$ in $F^1_{USCG}(\mathbb{R}^m)^p$, the properties (i) $U(\alpha)$ is bounded in $\mathbb{R}^m$ for each $\alpha \in (0, 1]$, and (ii) $U$ is uniformly $p$-mean bounded, are equivalent.

Proposition 3.9. Let $U$ be a subset of $F^1_{USCG}(X)^p$. If $U$ is $p$-mean equi-left-continuous, then for each $h \in [0, 1]$, there exists a $C_h \in \mathbb{R}$ such that for all $u \in U$,

$$
\left( \int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \leq C_h.
$$

Proof. Since $U$ is $p$-mean equi-left-continuous, then there is an $h_0 > 0$ such that for all $u \in U$ and $h \in [0, h_0]$,

$$
\left( \int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \leq 1. \tag{6}
$$

Let $h \in [0, 1]$. If $h \in [0, h_0]$, then the desired result follows from (6).

If $h \in (h_0, 1]$, then there is an $N(h) \in \mathbb{N}$ such that $h/N \leq h_0$. Thus for all $u \in U$,

$$
\left( \int_h^1 H([u]_\alpha, [u]_{\alpha-h})^p \, d\alpha \right)^{1/p} \\
\leq \sum_{k=0}^{N-1} \left( \int_h^{1-h/k} H([u]_\alpha, [u]_{\alpha-h/k})^p \, d\alpha \right)^{1/p} \\
\leq \sum_{k=0}^{N-1} \left( \int_{h-h/k}^{1-h/k} H([u]_\alpha, [u]_{\alpha-h/k})^p \, d\alpha \right)^{1/p} \\
\leq \sum_{k=0}^{N-1} \left( \int_{h/k}^{1-h/k} H([u]_\alpha, [u]_{\alpha-h/k})^p \, d\alpha \right)^{1/p} \\
\leq \sum_{k=0}^{N-1} \left( \int_{h/k}^{1-h/k} H([u]_\alpha, [u]_{\alpha-h/k})^p \, d\alpha \right)^{1/p} \\
\leq N.
$$

□
We can see that (i)⇒(iii) in the proof of Theorem 3.7 can also be proved by using Proposition 3.9.

- A subset $Y$ of a topological space $Z$ is said to be compact if for every set $I$ and every family of open sets, $O_i$, $i \in I$, such that $Y \subset \bigcup_{i \in I} O_i$ there exists a finite family $O_{i_1}, O_{i_2}, \ldots, O_{i_n}$ such that $Y \subseteq O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_n}$. In the case of a metric topology, the criterion for compactness becomes that any sequence in $Y$ has a subsequence convergent in $Y$.

- A relatively compact subset $Y$ of a topological space $Z$ is a subset with compact closure. In the case of a metric topology, the criterion for relative compactness becomes that any sequence in $Y$ has a subsequence convergent in $Z$.

- Let $(X,d)$ be a metric space. A set $U$ in $X$ is totally bounded if and only if, for each $\varepsilon > 0$, it contains a finite $\varepsilon$ approximation, where an $\varepsilon$ approximation to $U$ is a subset $S$ of $U$ such that $\rho(x,S) < \varepsilon$ for each $x \in U$.

Let $(X,d)$ be a metric space. A set $U$ is compact in $(X,d)$ implies that $U$ is relatively compact in $(X,d)$, which in turn implies that $U$ is totally bounded in $(X,d)$.

Suppose that $U$ is a subset of $F_{USCG}^1(X)$ and $\alpha \in [0,1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u \in U} [u]_\alpha$, and
- $U_\alpha := \{[u]_\alpha : u \in U\}$.

We list the following conclusions in [9] which are on the characterizations of total boundedness, relative compactness and compactness for $(F_{USCG}^1(X), H_{end})$ and the completion of $(F_{USCG}^1(X), H_{end})$. These conclusions will be useful in this paper.

**Theorem 3.10.** [9] Let $U$ be a subset of $F_{USCG}^1(X)$. Then $U$ is totally bounded in $(F_{USCG}^1(X), H_{end})$ if and only if $U(\alpha)$ is totally bounded in $(X,d)$ for each $\alpha \in (0,1]$.

**Theorem 3.11.** [9] Let $U$ be a subset of $F_{USCG}^1(X)$. Then $U$ is relatively compact in $(F_{USCG}^1(X), H_{end})$ if and only if $U(\alpha)$ is relatively compact in $(X,d)$ for each $\alpha \in (0,1]$.
Theorem 3.12. [9] Let $U$ be a subset of $F^1_{USCG}(X)$. Then the following are equivalent:

(i) $U$ is compact in $(F^1_{USCG}(X), H_{end})$;

(ii) $U(\alpha)$ is relatively compact in $(X, d)$ for each $\alpha \in (0, 1]$ and $U$ is closed in $(F^1_{USCG}(X), H_{end})$;

(iii) $U(\alpha)$ is compact in $(X, d)$ for each $\alpha \in (0, 1]$ and $U$ is closed in $(F^1_{USCG}(X), H_{end})$.

Theorem 3.13. [9] $(F^1_{USCG}(\tilde{X}), H_{end})$ is a completion of $(F^1_{USCG}(X), H_{end})$.

4. Characterizations of compactness in $(F^1_{USCG}(X)^p, d_p)$

In this section, we give the characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X)^p, d_p)$.

The relationship between the $d_p$ metric and the $H_{end}$ metric given in Theorem 3.1 (see the illustrations in Remark 3.2) will be used repeatedly in the sequel.

Lemma 4.1. If $u \in F^1_{USCG}(X)^p$, then $u$ is $p$-mean left-continuous.

Proof. The desired result can be proved in a similar fashion to Lemma 4.3 in [7] by replacing $\{0\}$ with $\{x_0\}$, where $0$ denotes the point $(0, \ldots, 0)$ in $\mathbb{R}^m$ and $x_0$ is a point in $X$.

Theorem 4.2. Let $U$ be a subset of $F^1_{USCG}(X)^p$. Then $U$ is a relatively compact set in $(F^1_{USCG}(X)^p, d_p)$ if and only if

(i) $U$ is a relatively compact set in $(F^1_{USCG}(X), H_{end})$, and

(ii) $U$ is $p$-mean equi-left-continuous.

Proof. Necessity. If $U$ is a relatively compact set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$, then by Theorem 3.1, (i) is true.

The necessity of (ii) can be proved in a similar fashion to the necessity of (ii) in Theorem 4.1 in [7] (Theorem 5.5).

Sufficiency. The proof is similar to the sufficiency part of Theorem 4.1 in [7]. A sketch of the proof is given as follows.
Let \( \{u_n\} \) be a sequence in \( U \). To find a subsequence \( \{v_n\} \) of \( \{u_n\} \) which converges to \( v \in F_{USCG}^1(X)^p \) according to the \( d_p \) metric, we split the proof into three steps.

**Step 1.** Find a subsequence \( \{v_n\} \) of \( \{u_n\} \) and \( v \in F_{USCG}^1(X)^p \) such that
\[
H_{\text{end}}(u_n, u) \to 0, \quad [0, 1] .
\] (7)
By (i) and Theorem 4.4 in [9], this step can be done immediately.

**Step 2.** Prove that
\[
\left( \int_0^1 H([v_n]_{\alpha}, [v]_{\alpha})^p \, d\alpha \right)^{1/p} \to 0 .
\] (8)
Proceeding according to the Step 2 in the sufficiency part of the proof of Theorem 4.1 in [7], we can obtain the desired result.

Here we mention one thing. In this proof of step 2, we need to prove the conclusion: for each \( h \in (0, 1] \),
\[
\left( \int_h^1 H([v_n]_{\alpha}, [v]_{\alpha})^p \, d\alpha \right)^{1/p} \to 0 .
\] (9)
In the proof of the corresponding conclusion in [7], there is a small mistake (or misprints). In the following, we give a slightly adjusted proof for the above conclusion (Obviously, the proof of the corresponding conclusion in [7] can be adjusted similarly).

Note that \( [v_n]_h \) and \( [v]_h \) are contained in \( \overline{U(h/2)} \), which is compact in \( X \) according to Theorem 3.11. Then there is an \( M(h) \geq 0 \) such that
\[
\max\{d(x, y) : x, y \in \overline{U(h/2)}\} \leq M(h) .
\]
Hence
\[
H([v_n]_{\alpha}, [v]_{\alpha}) \leq M(h)
\]\nfor \( \alpha \in [h, 1] \) and \( n = 1, 2, \ldots \). Combined with (7) and by using the Lebesgue’s dominated convergence theorem, we thus obtain (9).

**Step 3.** Show that \( v \in F_{USCG}^1(X)^p \).

Since \( v \in F_{USCG}^1(X) \), it suffices to show that
\[
\left( \int_0^1 H([v]_{\alpha}, \{x_0\})^p \, d\alpha \right)^{1/p} < +\infty \quad \text{for some} \quad x_0 \in X,
\]
which can be proved in a similar fashion to the
conclusion \( \left( \int_0^1 H([v]_\alpha, \{0\})^p \, d\alpha \right)^{1/p} < +\infty \) in the Step 3 in the sufficiency part of Theorem 4.1 in [7] by replacing \( \{0\} \) with \( \{x_0\} \), where \( x_0 \in X \).

Theorem 4.3. Let \( U \) be a subset of \( F^1_{USCG}(X)^p \). Then \( U \) is a relatively compact set in \( (F^1_{USCG}(X)^p, d_p) \) if and only if

(i) \( U(\alpha) \) is relatively compact in \( (X, d) \) for each \( \alpha \in (0, 1] \), and

(ii) \( U \) is \( p \)-mean equi-left-continuous.

Proof. The desired result follows immediately from Theorems 3.11 and 4.2.

Theorem 4.4. Let \( U \) be a subset of \( F^1_{USCG}(X)^p \). Then \( U \) is a totally bounded set in \( (F^1_{USCG}(X)^p, d_p) \) if and only if

(i) \( U \) is a totally bounded set in \( (\tilde{F}_{USCG}(X), H_{end}) \), and

(ii) \( U \) is \( p \)-mean equi-left-continuous.

Proof. Necessity. Suppose that \( U \) is totally bounded in \( (F^1_{USCG}(X)^p, d_p) \). Then by (2), \( U \) is a totally bounded set in \( (\tilde{F}_{USCG}(X), H_{end}) \); that is, (i) is true.

The necessity of (ii) can be proved in a similar fashion to the necessity of (ii) in Theorem 4.1 in [7], which is Theorem 5.5 in this paper.

Sufficiency. Suppose that \( U \) satisfies (i) and (ii). From Theorem 3.13, \( U \) is a relatively compact set in \( (\tilde{F}_{USCG}(\tilde{X}), H_{end}) \). Then, by Theorem 4.2, \( U \) is a relatively compact set in \( (F^1_{USCG}(\tilde{X})^p, d_p) \), and thus \( U \) is a totally bounded set in \( (F^1_{USCG}(X)^p, d_p) \).

Theorem 4.5. Let \( U \) be a subset of \( F^1_{USCG}(X)^p \). Then \( U \) is a totally bounded set in \( (F^1_{USCG}(X)^p, d_p) \) if and only if

(i) \( U(\alpha) \) is totally bounded in \( (X, d) \) for each \( \alpha \in (0, 1] \), and

(ii) \( U \) is \( p \)-mean equi-left-continuous.

Proof. The desired result follows immediately from Theorems 3.10 and 4.4.

Theorem 4.6. Let \( U \) be a subset of \( F^1_{USCG}(X)^p \). Then \( U \) is compact in \( (F^1_{USCG}(X)^p, d_p) \) if and only if

(i) \( U(\alpha) \) is relatively compact in \( (X, d) \) for each \( \alpha \in (0, 1] \),

(ii) \( U \) is \( p \)-mean equi-left-continuous, and

(iii) \( U \) is a closed set in \( (F^1_{USCG}(X)^p, d_p) \).
Proof. The desired result follows immediately from Theorem 4.3.

Theorem 4.7. Let $U$ be a subset of $F_{USCG}^1(X)^p$. Then $U$ is compact in $(F_{USCG}^1(X)^p, d_p)$ if and only if

(i) $U(\alpha)$ is compact in $(X, d)$ for each $\alpha \in (0, 1]$,

(ii) $U$ is $p$-mean equi-left-continuous, and

(iii) $U$ is a closed set in $(F_{USCG}^1(X)^p, d_p)$.

Proof. By Theorem 4.6, to show the desired result, we only need to show that if $U$ is compact in $(F_{USCG}^1(X)^p, d_p)$, then (i) is true.

Assume that $U$ is compact in $(F_{USCG}^1(X)^p, d_p)$, then by (2), $U$ is compact in $(F_{USCG}^1(X), H_{\text{end}})$, hence from Theorem 3.12, (i) is true.

Theorem 4.8. Let $U$ be a subset of $F_{USCG}^1(X)^p$. Then $U$ is a compact set in $(F_{USCG}^1(X)^p, d_p)$ if and only if

(i) $U$ is a compact set in $(F_{USCG}^1(X), H_{\text{end}})$, and

(ii) $U$ is $p$-mean equi-left-continuous.

Proof. Necessity. Assume that $U$ is compact in $(F_{USCG}^1(X)^p, d_p)$. Hence by Theorem 3.1, $U$ is compact in $(F_{USCG}^1(X), H_{\text{end}})$, i.e. (i) is true. By Theorem 4.7, (ii) is true.

Sufficiency. Assume that (i) and (ii) are true. Then by Theorem 4.2, $U$ is relatively compact in $(F_{USCG}^1(X)^p, d_p)$. To show that $U$ is compact in $(F_{USCG}^1(X)^p, d_p)$, we only need to show that $U$ is closed in $(F_{USCG}^1(X)^p, d_p)$.

To do this, let $\{u_n\}$ be a sequence in $U$ and $d_p(u_n, u) \to 0$. Then by Theorem 3.1, $H_{\text{end}}(u_n, u) \to 0$. Since from (i), $U$ is closed in $(F_{USCG}^1(X), H_{\text{end}})$, we have that $u \in U$. So $U$ is closed in $(F_{USCG}^1(X)^p, d_p)$.

From Theorems 4.2, 4.4 and 4.8, we obtain the following conclusion:

- Let $U$ be a subset in $F_{USCG}^1(X)^p$. Then $U$ is total bounded (respectively, relatively compact, compact) in $(F_{USCG}^1(X)^p, d_p)$ if and only if $U$ is total bounded (respectively, relatively compact, compact) in $(F_{USCG}^1(X), H_{\text{end}})$ and $U$ is $p$-mean equi-left-continuous.
5. Characterizations of compactness in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\)

In this section, we discuss the characterizations of total boundedness, relative compactness and compactness in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\). We point out that the conclusions on the characterizations of total boundedness, relative compactness and compactness in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\) given in our previous work [10] can be seen as corollaries of the corresponding results for \((F_{\text{USCG}}^1(X)^p, d_p)\) given in Section 4 of this paper. Furthermore, by using results in Sections 3 and 4, we give new characterizations of total boundedness, relative compactness and compactness in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\).

Note that for a subset \(V\) of \(\mathbb{R}^m\), the conditions (i) \(V\) is relatively compact in \(\mathbb{R}^m\), (ii) \(V\) is totally bounded in \(\mathbb{R}^m\), and (iii) \(V\) is bounded in \(\mathbb{R}^m\), are equivalent to each other. Thus Theorems 4.3, 4.5, 4.6 and 4.7 imply the following four conclusions on the characterizations of compactness in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\), respectively.

**Corollary 5.1.** Let \(U\) be a subset of \(F_{\text{USCG}}^1(\mathbb{R}^m)^p\). Then \(U\) is a relatively compact set in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\) if and only if
(i) \(U(\alpha)\) is bounded in \(\mathbb{R}^m\) for each \(\alpha \in (0, 1]\), and
(ii) \(U\) is \(p\)-mean equi-left-continuous.

**Corollary 5.2.** Let \(U\) be a subset of \(F_{\text{USCG}}^1(\mathbb{R}^m)^p\). Then \(U\) is a totally bounded set in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\) if and only if
(i) \(U(\alpha)\) is bounded in \(\mathbb{R}^m\) for each \(\alpha \in (0, 1]\), and
(ii) \(U\) is \(p\)-mean equi-left-continuous.

**Corollary 5.3.** Let \(U\) be a subset of \(F_{\text{USCG}}^1(\mathbb{R}^m)^p\). Then \(U\) is compact in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\) if and only if
(i) \(U(\alpha)\) is bounded in \(\mathbb{R}^m\) for each \(\alpha \in (0, 1]\),
(ii) \(U\) is \(p\)-mean equi-left-continuous, and
(iii) \(U\) is a closed set in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\).

**Corollary 5.4.** Let \(U\) be a subset of \(F_{\text{USCG}}^1(\mathbb{R}^m)^p\). Then \(U\) is compact in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\) if and only if
(i) \(U(\alpha)\) is compact in \(\mathbb{R}^m\) for each \(\alpha \in (0, 1]\),
(ii) \(U\) is \(p\)-mean equi-left-continuous, and
(iii) \(U\) is a closed set in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\).

In [7], we have obtained the following three conclusions on the characterizations of compactness in \((F_{\text{USCG}}^1(\mathbb{R}^m)^p, d_p)\).
Theorem 5.5. (Theorem 4.1 in [7]) Let $U$ be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then $U$ is a relatively compact set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if
(i) $U$ is uniformly $p$-mean bounded, and
(ii) $U$ is $p$-mean equi-left-continuous.

Theorem 5.6. (Theorem 4.2 in [7]) Let $U$ be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then $U$ is a totally bounded set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if
(i) $U$ is uniformly $p$-mean bounded, and
(ii) $U$ is $p$-mean equi-left-continuous.

Theorem 5.7. (Theorem 4.3 in [7]) Let $U$ be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then $U$ is compact in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if
(i) $U$ is uniformly $p$-mean bounded,
(ii) $U$ is $p$-mean equi-left-continuous, and
(iii) $U$ is a closed set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$.

Remark 5.8. From Theorem 3.7, we can see:
Corollary 5.1 implies Theorem 4.1 in [7] (which is Theorem 5.5 in this paper), and the converse is true;
Corollary 5.2 implies Theorem 4.2 in [7] (which is Theorem 5.6 in this paper), and the converse is true;
Corollary 5.3 implies Theorem 4.3 in [7] (which is Theorem 5.7 in this paper), and the converse is true.

So the characterizations of compactness for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ in [7] (Theorems 4.1, 4.2 and 4.3 in [7]) can be seen as corollaries of the characterizations of compactness for $(F^1_{USCG}(X)^p, d_p)$ in this paper (Theorems 4.3, 4.5 and 4.6).
The results of the characterizations of compactness for $(F^1_{USCG}(X)^p, d_p)$ in this paper generalize the corresponding results for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ in [7].

In the following, we give new characterizations of compactness for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$.

Using Theorem 3.7, we can obtain the following characterizations of compactness for $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ from Corollaries 5.1, 5.2 and 5.3.

Theorem 5.9. Let $U$ be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then $U$ is a relatively compact set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if
(i) There exists an $h \in (0, 1)$ such that $U(h)$ is bounded in $\mathbb{R}^m$, and
(ii) $U$ is $p$-mean equi-left-continuous.

Theorem 5.10. Let $U$ be a subset of $F^1_{USCG}(\mathbb{R}^m)^p$. Then $U$ is a totally bounded set in $(F^1_{USCG}(\mathbb{R}^m)^p, d_p)$ if and only if
Corollary 6.2. \((F_{USCG}^1(\mathbb{R}^m), d_p)\) is complete.

Proof. Since \(\mathbb{R}^m\) is complete, the desired result follows immediately from Theorem 6.1. \(\square\)
Remark 6.3. Corollary 6.2 is Theorem 5.1 in [7]. So Theorem 6.1 in this paper generalizes Theorem 5.1 in [7].

For \( u \in F^1_{USCG}(X) \) and \( \varepsilon > 0 \), define \( u^\varepsilon \in F^1_{USCB}(X) \) by
\[
[u^\varepsilon]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (\varepsilon, 1], \\ [u]_\varepsilon, & \alpha \in [0, \varepsilon]. \end{cases}
\]

Theorem 6.4. \( F^1_{USCB}(X) \) is a dense set in \( (F^1_{USCG}(X)^p, d_p) \).

Proof. The desired result can be proved in a similar fashion to Theorem 5.2 in [7]. In fact, it is shown that for each \( v \in F^1_{USCG}(X)^p \), \( d_p(v^{(1/n)}, v) \to 0 \).

Theorem 6.5. \( (F^1_{USCG}(\tilde{X})^p, d_p) \) is a completion of \( (F^1_{USCB}(X), d_p) \).

Proof. In the proof of Theorem 6.3 in [9], we show the following conclusion

- for each \( v \in F^1_{USCB}(\tilde{X}) \) and each \( \varepsilon > 0 \), there is a \( w \in F^1_{USCB}(X) \) such that \( H([w]_\alpha, [v]_\alpha) \leq \varepsilon \) for all \( \alpha \in [0, 1] \).

Thus we have \( d_p(v, w) \leq \varepsilon \). This means that \( F^1_{USCB}(X) \) is dense in \( (F^1_{USCB}(\tilde{X}), d_p) \).

From Theorem 6.4, we know that \( F^1_{USCB}(\tilde{X}) \) is dense in \( (F^1_{USCG}(\tilde{X})^p, d_p) \).

Combined the above conclusions, we obtain that \( F^1_{USCB}(X) \) is dense in \( (F^1_{USCG}(\tilde{X})^p, d_p) \). By Theorem 6.1, \( (F^1_{USCG}(\tilde{X})^p, d_p) \) is complete. So \( (F^1_{USCG}(\tilde{X})^p, d_p) \) is a completion of \( (F^1_{USCB}(X), d_p) \).

Corollary 6.6. \( (F^1_{USCG}(\tilde{X})^p, d_p) \) is a completion of \( (F^1_{USCG}(X)^p, d_p) \).

Proof. Since \( F^1_{USCB}(X) \subseteq F^1_{USCG}(X)^p \subseteq F^1_{USCG}(\tilde{X})^p \), the desired result follows immediately from Theorem 6.5.

7. Measurability of function \( H([u]_\alpha, [v]_\alpha) \)

For \( u, v \in F^1_{USC}(X) \), \( d_p(u, v) \) is well-defined if and only if \( H([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \( [0, 1] \). So it is important to discuss the measurability of the function \( H([u]_\alpha, [v]_\alpha) \) of \( \alpha \) on \( [0, 1] \). In this section, we give some fundamental conclusions on this topic.
We uniformly use $H$ to denote the Hausdorff metric on $C(X)$ induced by $d_X$, where $(X, d_X)$ is a certain metric space. The meaning of $H$ can be judged according to the context.

We have pointed out the following statements on the measurability of the function $H([u]_\alpha, [v]_\alpha)$ (See [8] or [10] which was submitted on 2019.07.06).

- For $u \in F_{USC}^1(X)$ and $x_0 \in X$, $H([u]_\alpha, \{x_0\})$ is a measurable function of $\alpha$ on $[0, 1]$.
- For $u, v \in F_{USC}^1(\mathbb{R}^m)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0, 1]$.
- For $u, v \in F_{USCG}^1(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0, 1]$.
- There exists a metric space $X$ and $u, v \in F_{USC}^1(X)$ such that $H([u]_\alpha, [v]_\alpha)$ is a non-measurable function of $\alpha$ on $[0, 1]$.

In this section, we first give the proofs of the first three statements and the example to show the last statement (we submitted the proofs of the first three statements in [11]). Then we give some improvements of these statements.

**Proposition 7.1.** For $u \in F_{USC}^1(X)$ and $x_0 \in X$, $H([u]_\alpha, \{x_0\})$ is a measurable function of $\alpha$ on $[0, 1]$.

**Proof.** We can see that for $0 \leq \alpha \leq \beta \leq 1$,

$$H([u]_\alpha, \{x_0\}) = \sup_{x \in [u]_\alpha} d(x, x_0) \geq \sup_{x \in [u]_\beta} d(x, x_0) = H([u]_\beta, \{x_0\}).$$

The desired result follows from the fact that $H([u]_\alpha, \{x_0\})$ is a monotone function of $\alpha$ on $[0, 1]$.

For $u, v \in F_{USC}^1(X)$ and $r \in \mathbb{R}$, we use the symbol $\{H(u, v) > r\}$ to denote the set $\{\alpha \in [0, 1] : H([u]_\alpha, [v]_\alpha) > r\}$.

**Proposition 7.2.** For $u, v \in F_{USC}^1(\mathbb{R}^m)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0, 1]$. 

22
Proof. We only need to show that for each $r \in \mathbb{R}$, the set $\{H(u, v) > r\}$ is a measurable set.

Step (i) For each $r \in \mathbb{R}$, if $\alpha > 0$ and $\alpha \in \{H(u, v) > r\}$, then there exists $\delta(\alpha) > 0$ such that $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\}$.

We proceed by contradiction. If for each $\delta > 0$, $[\alpha - \delta, \alpha] \not\subseteq \{H(u, v) > r\}$, then there exists an increasing sequence $\{\gamma_n\}$ such that $\gamma_n \to \alpha$ and

$$H([u]_{\gamma_n}, [v]_{\gamma_n}) \leq r.$$  

Given $x \in [u]_{\alpha}$, then $d(x, [v]_{\gamma_n}) \leq H([u]_{\gamma_n}, [v]_{\gamma_n}) \leq r$. Therefore there exist $y_n \in [v]_{\gamma_n}$ such that $d(x, y_n) = d(x, [v]_{\gamma_n}) \leq r$. Hence there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{y_{n_i}\}$ converges to $y \in \mathbb{R}^m$. Note that $d(x, y) \leq r$ and $y \in \cap[v]_{\gamma_{n_i}} = [v]_{\alpha}$, so we have $d(x, [v]_{\alpha}) \leq r$.

From the arbitrariness of $x$, $H^*([v]_{\alpha}, [u]_{\alpha}) \leq r$. Similarly, we can deduce that $H^*([v]_{\alpha}, [u]_{\alpha}) \leq r$. Thus $H([u]_{\alpha}, [v]_{\alpha}) \leq r$, which is a contradiction.

Step (ii) For each $r \in \mathbb{R}$, if $\{H(u, v) > r\} \setminus \{0\} \neq \emptyset$, then $\{H(u, v) > r\} \setminus \{0\}$ is a union of disjoint positive length intervals.

Suppose that $\{H(u, v) > r\} \setminus \{0\} \neq \emptyset$. For $x \in \{H(u, v) > r\} \setminus \{0\}$, let $\widehat{x} = \bigcup\{[a, b] : x \in [a, b] \subseteq \{H(u, v) > r\} \setminus \{0\}\}$, i.e., $\widehat{x}$ is the largest interval in $\{H(u, v) > r\} \setminus \{0\}$ which contains $x$. Then by step (i), $\widehat{x}$ is a positive length interval. Note that for $x, y \in \{H(u, v) > r\} \setminus \{0\}$, if $\widehat{x} \cap \widehat{y} \neq \emptyset$, then $\widehat{x} = \widehat{y}$. Thus $\{H(u, v) > r\} \setminus \{0\}$ is a union of disjoint positive length intervals.

Step (iii) For each $r \in \mathbb{R}$, $\{H(u, v) > r\}$ is a measurable set.

Clearly, if positive length intervals are disjoint, then these positive length intervals are at most countable. Thus, from step (ii), $\{H(u, v) > r\} \setminus \{0\}$ is a measurable set. So $\{H(u, v) > r\}$ is a measurable set.

\[\square\]

Remark 7.3. Let $u, v \in F^1_{U\text{SC}}(X)$. For each $r \in \mathbb{R}$, if $0 \in \{H(u, v) > r\}$, then there exists $\delta > 0$ such that $[0, \delta] \subseteq \{H(u, v) > r\}$.

The above fact is equivalent to the following fact

Let $u, v \in F^1_{U\text{SC}}(X)$. Then $H([u]_0, [v]_0) \leq \liminf_{\alpha \to 0^+} H([u]_{\alpha}, [v]_{\alpha})$, where $\liminf_{\alpha \to 0^+} H([u]_{\alpha}, [v]_{\alpha}) = +\infty$ is possible.

Combined this fact with the proof of Proposition 7.2, we have the following conclusion

23
Let \( u, v \in F_{USC}^{1}(\mathbb{R}^m) \) and let \( r \in \mathbb{R} \). If \( \{H(u, v) > r\} \neq \emptyset \), then 
\( \{H(u, v) > r\} \) is a union of disjoint positive length intervals (Obviously, 
\( \{H(u, v) > r\} \) could be an interval. It is easy to see that for fixed \( r \geq 0 \), 
the possible forms of the maximal intervals in \( \{H(u, v) > r\} \) are as follows: 
\[ [0, \alpha), [0, \alpha], (\beta, \alpha) \) and \( (\beta, \alpha) \), where \( \alpha \in (0, 1] \) and \( \beta \in [0, 1] \)).

For \( f(\alpha) : [0, 1] \to [0, +\infty) \cup \{+\infty\} \) and \( \alpha \in (0, 1] \), 
\( \lim \inf_{\gamma \to \alpha-} f(\alpha) \) is defined by 
\( \lim \inf_{\gamma \to \alpha-} f(\alpha) := \inf\{x \in [0, +\infty) \cup \{+\infty\} : \text{there is a sequence } \{\gamma_n\} \text{ such that } \gamma_n \to \alpha - \text{ and } x = \lim_{n \to +\infty} f(\gamma_n)\} \). For \( f(\alpha) : [0, 1] \to [0, +\infty) \cup \{+\infty\} \) and 
\( \alpha \in (0, 1) \), \( \lim \inf_{\gamma \to \alpha-} f(\alpha) \) exists and \( \lim \inf_{\gamma \to \alpha-} f(\alpha) = +\infty \) is possible.

We can check that \( \lim \inf_{\gamma \to \alpha-} f(\alpha) = \min\{x \in [0, +\infty) \cup \{+\infty\} : \text{there is a sequence } \{\gamma_n\} \text{ such that } \gamma_n \to \alpha - \text{ and } x = \lim_{n \to +\infty} f(\gamma_n)\} \). Clearly if \( \lim \inf_{\gamma \to \alpha-} f(\alpha) \) exists, then \( \lim \inf_{\gamma \to \alpha-} f(\alpha) = \lim \inf_{\gamma \to \alpha-} f(\alpha) \).

**Remark 7.4.** Let \( u, v \in F_{USC}^{1}(X) \) and let \( \alpha > 0 \). Then the following properties (i) and (ii) are equivalent.

(i) For each \( r \in \mathbb{R} \), if \( \alpha \in \{H(u, v) > r\} \), then there exists \( \delta(\alpha) > 0 \) such that 
\( [\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\} \).

(ii) \( H([u]_{\alpha}, [v]_{\alpha}) \leq \lim \inf_{\gamma \to \alpha-} H([u]_{\gamma}, [v]_{\gamma}) \) (\( \lim \inf_{\gamma \to \alpha-} H([u]_{\gamma}, [v]_{\gamma}) = +\infty \) is possible).

So for \( u, v \in F_{USC}^{1}(X) \), the property \( "H([u]_{\alpha}, [v]_{\alpha}) \leq \lim \inf_{\gamma \to \alpha-} H([u]_{\gamma}, [v]_{\gamma}) \) 
for all \( \alpha \in (0, 1)" \) is equivalent to the property listed below, which is given as the conclusion of step (i) of the proof of Proposition 7.2:

- For each \( r \in \mathbb{R} \), if \( \alpha > 0 \) and \( \alpha \in \{H(u, v) > r\} \), then there exists 
\( \delta(\alpha) > 0 \) such that \( [\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\} \).

The equivalence of properties (i) and (ii) given at the beginning of this remark follows from basic analysis. Assume that (i) is true. Given \( r \in \mathbb{R} \) 
with \( H([u]_{\alpha}, [v]_{\alpha}) > r \). Then there exists \( \delta(\alpha) > 0 \) such that 
\( [\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\} \), thus \( \lim \inf_{\gamma \to \alpha-} H([u]_{\gamma}, [v]_{\gamma}) \geq r \). From the arbitrariness of 
\( r \in (-\infty, H([u]_{\alpha}, [v]_{\alpha})) \), we have \( H([u]_{\alpha}, [v]_{\alpha}) \leq \lim \inf_{\gamma \to \alpha-} H([u]_{\gamma}, [v]_{\gamma}) \). So 
(ii) is true.

Assume that (ii) is true. Let \( \alpha > 0 \) and \( r \in \mathbb{R} \). If \( \alpha \in \{H(u, v) > r\} \), then \( \lim \inf_{\gamma \to \alpha-} H([u]_{\gamma}, [v]_{\gamma}) > r \). We claim that there exists \( \delta(\alpha) > 0 \) such that 
\( [\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\} \). Otherwise there exists \( \{\alpha_n\} \) in \( [0, 1] \) such that 
\( \alpha_n \to \alpha - \) \( \text{and } H([u]_{\alpha_n}, [v]_{\alpha_n}) \leq r \), which contradicts \( \lim \inf_{\gamma \to \alpha-} H([u]_{\gamma}, [v]_{\gamma}) > r \).
Remark 7.5. We can see that step (i) in the proof of Proposition 7.2 shows the following statement

(a) For \( u, v \in F^1_{USC}(\mathbb{R}^m) \), \( u, v \) satisfy the property which is given as the conclusion of step (i) in the proof of Proposition 7.2.

We can see that steps (ii) and (iii) in the proof of Proposition 7.2 show the following statement

(b) For \( u, v \in F^1_{USC}(X) \), if \( u, v \) satisfy the property which is given as the conclusion of step (i) in the proof of Proposition 7.2, then \( H([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\).

So from the proof of Proposition 7.2 and Remark 7.4, we know that

(c) For \( u, v \in F^1_{USC}(X) \), if \( H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha^-} H([u]_\gamma, [v]_\gamma) \) for all \( \alpha \in (0, 1] \), then \( H([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\).

(a') Let \( u, v \in F^1_{USC}(\mathbb{R}^m) \). Then \( H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \to \alpha^-} H([u]_\gamma, [v]_\gamma) \) for all \( \alpha \in (0, 1] \).

The following Proposition 7.6 is Lemma 4.4 in [8].

Proposition 7.6. [8] Let \( U_n \in K(X) \) for \( n = 1, 2, \ldots \).

(i) If \( U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots \), then \( U = \bigcap_{n=1}^{+\infty} U_n \in K(X) \) and \( H(U_n, U) \to 0 \) as \( n \to +\infty \).

(ii) If \( U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq \ldots \) and \( V = \bigcup_{n=1}^{+\infty} U_n \in K(X) \), then \( H(U_n, V) \to 0 \) as \( n \to +\infty \).

Proof. (i) is easy to show. We only prove (ii). Suppose that \( H(U_n, U) \not\to 0 \). Then there is an \( \varepsilon_0 > 0 \) such that \( H(U_n, U) > \varepsilon_0 \). Hence there exists \( x_n \in U \) such that

\[
d(x_n, U_n) > \varepsilon_0.
\]

Since \( U \) is compact, there is a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( x_{n_i} \) converges to \( x \in U \). Note that there exists \( \{y_n\} \) such that \( y_n \in U_n \) and \( y_n \to x \). Thus \( d(x, U_n) \to 0 \), which contradicts (10).

From Proposition 7.6, we know that

Proposition 7.7. For \( u, v \in F^1_{USCG}(X) \), \( H([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\).
Proof. Note that $H([u]_\alpha, [v]_\alpha)$ is finite at $\alpha \in (0, 1]$ and for $\alpha, \beta \in (0, 1]$,

$$|H([u]_\alpha, [v]_\alpha) - H([u]_\beta, [v]_\beta)| \leq H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta).$$

Then by Proposition 7.6 (i), for each $\alpha \in (0, 1]$,

$$\lim_{\beta \to \alpha^-} H([u]_\beta, [v]_\beta) = H([u]_\alpha, [v]_\alpha),$$

i.e. $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$.

Thus from clause (c) in Remark 7.5, we have $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0, 1]$.

The desired result can also be shown in the following way. Since $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$, then obviously $u, v$ satisfy the property which is given as the conclusion of step (i), and thus by clause (b) in Remark 7.5, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0, 1]$.

\[ \blacksquare \]

Remark 7.8. From Proposition 7.6, we know that for $u, v \in F^{1}_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$, and that for $u, v \in F^{1}_{USCB}(X)$, $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$.

The first conclusion mentioned above has been shown in the proof of Proposition 7.7. The following proof of the last conclusion is similar as that of the first conclusion. Since for $u, v \in F^{1}_{USCB}(X)$ and $\alpha \in (0, 1]$,

$$|H([u]_\alpha, [v]_\alpha) - H([u]_0, [v]_0)| \leq H([u]_\alpha, [u]_0) + H([v]_\alpha, [v]_0).$$

Thus by Proposition 7.6 (ii), $\lim_{\alpha \to 0^+} H([u]_\alpha, [v]_\alpha) = H([u]_0, [v]_0)$, i.e. $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$.

Remark 7.9. In [10] (Lemma 6.3) and [8] (Lemma 6.5), we pointed out that for $u \in F^{1}_{UUSCG}(X)$, the cut-function $[u](\alpha) = [u]_\alpha$ from $[0, 1]$ to $(C(X), H)$ is left-continuous on $(0, 1]$. Then it follows immediately that for $u, v \in F^{1}_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is left-continuous at $\alpha \in (0, 1]$ (see Proposition 7.7). From this fact, it’s natural to realize that for $u, v \in F^{1}_{USCG}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0, 1]$.

We suspect the conclusion that a left continuous function $f : (0, 1] \to \mathbb{R}$ is measurable is an already known conclusion, although we can’t find this conclusion in the references that we can obtain. So in some sense, Proposition 7.7 can be seen as a corollary of Proposition 7.6.
Let \((X, d_X)\) be a metric space. We say that \(S \subseteq F_{USC}(X)\) satisfies condition \((X, d_X)\)-I if \([u]\alpha \cap B(x, r)\) is compact in \((X, d_X)\) for all \(u \in S\), \(\alpha \in (0, 1]\), \(x \in X\) and \(r \in \mathbb{R}^+\), where \(B(x, r) := \{y \in X : d_X(x, y) \leq r\}\).

Clearly, \(S = F_{USC}^1(\mathbb{R}^m)\) satisfies condition \(\mathbb{R}^m\)-I and \(S = F_{USCG}^1(X)\) satisfies condition \((X, d_X)\)-I.

If \(S \subseteq F_{USC}(X)\) satisfies condition \((X, d_X)\)-I, then proceed similarly as the step (i) of the proof of Proposition 7.2, we have that \(H([u]\alpha, [v]\alpha) \leq \lim \inf_{\gamma \to \alpha} H([u]_{\gamma}, [v]_{\gamma})\) for all \(u, v \in S\) and \(\alpha \in (0, 1]\). Thus as mentioned in Remark 7.5, for all \(u, v \in S\), \(H([u]\alpha, [v]\alpha)\) is a measurable function of \(\alpha\) on \([0, 1]\).

There exists metric space \((X, d_X)\) and \(S \subseteq F_{USC}(X)\) which satisfies a condition weaker than condition \((X, d_X)\)-I. By using this weaker condition, we can proceed similarly as the step (i) of the proof of Proposition 7.2 to show that \(H([u]\alpha, [v]\alpha) \leq \lim \inf_{\gamma \to \alpha} H([u]_{\gamma}, [v]_{\gamma})\) for all \(u, v \in S\) and \(\alpha \in (0, 1]\).

To give the example which shows the last statement presented in [8] which is listed at the beginning of this section, we need some conclusions at first.

The following representation theorem should be a known conclusion.

**Theorem 7.10.** Let \(X\) be a set. Given \(u \in F(X)\), then for all \(\alpha \in (0, 1]\), \([u]\alpha = \cap_{\beta < \alpha} [u]_{\beta}\).

Conversely, suppose that \(\{u(\alpha) : \alpha \in (0, 1]\}\) is a family of sets in \(X\) satisfying \(u(\alpha) = \cap_{\beta < \alpha} u(\beta)\) for all \(\alpha \in (0, 1]\). Define \(v \in F(X)\) by \(v(x) := \sup\{\alpha : x \in u(\alpha)\}\) (\(\sup \emptyset = 0\)). Then \([v]\alpha = u(\alpha)\) for all \(\alpha \in (0, 1]\).

**Proof.** (The proof is routine.) Let \(u \in F(X)\) and \(\alpha \in (0, 1]\). For each \(x \in X\), \(x \in [u]\alpha \iff u(x) \geq \alpha \iff \) for each \(\beta < \alpha\), \(u(x) \geq \beta \iff \) for each \(\beta < \alpha\), \(x \in [u]_{\beta}\). So \([u]\alpha = \cap_{\beta < \alpha} [u]_{\beta}\).

Conversely, suppose that \(\{u(\alpha) : \alpha \in (0, 1]\}\) is a family of sets in \(X\) satisfying \(u(\alpha) = \cap_{\beta < \alpha} u(\beta)\) for all \(\alpha \in (0, 1]\). Define \(v \in F(X)\) by \(v(x) := \sup\{\alpha : x \in u(\alpha)\}\) (\(\sup \emptyset = 0\)). To show \([v]\alpha = u(\alpha)\) for all \(\alpha \in (0, 1]\), it suffices to show that for each \(\alpha \in (0, 1]\), \([v]\alpha \supseteq u(\alpha)\) and \([v]\alpha \subseteq u(\alpha)\).

Let \(\alpha \in (0, 1]\). For each \(x \in X\), if \(x \in u(\alpha)\), then clearly \(v(x) \geq \alpha\), i.e. \(x \in [v]\alpha\). So \([v]\alpha \supseteq u(\alpha)\).

For each \(x \in X\), if \(x \in [v]\alpha\), i.e. \(v(x) \geq \alpha\). Then \(\sup\{\beta : x \in u(\beta)\} \geq \alpha\), hence for \(n = 1, 2, \ldots\), there exists \(\beta_n\) such that \(1 \geq \beta_n \geq \alpha - 1/n\) and \(x \in u(\beta_n)\). Set \(\gamma = \sup_{n=1}^{+\infty} \beta_n\). Then \(1 \geq \gamma \geq \alpha\) and thus \(x \in \cap_{n=1}^{+\infty} u(\beta_n) = u(\gamma) \subseteq u(\alpha)\). So \([v]\alpha \subseteq u(\alpha)\). \(\square\)
Let \((Y, \rho)\) be an extended metric space. For \(y \in Y\) and \(\varepsilon > 0\), let \(B(y, \varepsilon)\) denote the set \(\{z \in Y : \rho(y, z) < \varepsilon\}\). \(\{B(y, \varepsilon) : y \in Y, \varepsilon > 0\}\) is a basis for the topology induced by \(\rho\) on \(Y\). The closure of a set \(A\) in \((Y, \rho)\), denoted by \(\overline{A}\), refers to the closure of \(A\) in \(Y\) according to the topology induced by \(\rho\) on \(Y\). Then \(x \in \overline{A}\) if and only if there is a sequence \(\{x_n\}\) in \(Y\) such that \(\rho(x_n, x) \to 0\). So \(x \in \overline{A}\) if and only if \(\rho(x, A) = 0\).

Here we mention that if \((Y, \rho)\) is an extended metric space, then the Hausdorff distance \(H\) on \(C(Y)\) induced by \(\rho\) using (1) is an extended metric on \(C(Y)\), where \(C(Y)\) denotes the set of nonempty closed sets in \((Y, \rho)\). It can be seen that \(H\) satisfies positivity and symmetry. To show that \(H\) satisfies the triangle inequality, we only need to show that

\[H^*(U, W) \leq H^*(U, V) + H^*(V, W)\]  

(11)

for \(U, V, W \in C(Y)\). To do this, let \(x \in U\). Then

\[\rho(x, W) \leq \inf_{y \in V} \inf_{z \in W} \{\rho(x, y) + \rho(y, z)\}\]
\[\leq \inf_{y \in V} \{\rho(x, y) + \rho(y, W)\}\]
\[\leq \inf_{y \in V} \rho(x, y) + H^*(V, W)\]
\[= \rho(x, V) + H^*(V, W)\]
\[\leq \rho(x, V) + H^*(V, W)\]
\[\leq H^*(U, V) + H^*(V, W).\]

From the arbitrariness of \(x\) in \(U\), we obtain (11). So the Hausdorff distance \(H\) on \(C(Y)\) is the Hausdorff extended metric.

For simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric in this paper.

For an extended metric space \((Y, \rho)\), we define

\[F_{U, SC}(Y) = \{u \in F(Y) : [u]_{\alpha} \text{ is closed in } (Y, \rho) \text{ for } \alpha \in (0, 1]\}\].

Let \(\Gamma\) be a set, and for each \(\gamma \in \Gamma\), let \((X_\gamma, d_\gamma)\) be a metric space. Define an extended metric \(d\) on \(\prod_{\gamma \in \Gamma} X_\gamma\) as

\[d(x, y) := \sup\{d_\gamma(x_\gamma, y_\gamma) : \gamma \in \Gamma\}\]  

(12)

for \(x = (x_\gamma)_{\gamma \in \Gamma}\) and \(y = (y_\gamma)_{\gamma \in \Gamma}\).

We use the symbol \(\prod_{\gamma \in \Gamma} (X_\gamma, d_\gamma)\) to denote the extended metric space \((\prod_{\gamma \in \Gamma} X_\gamma, d)\). If not mentioned specially, we suppose by default that the extended metric on \(\prod_{\gamma \in \Gamma} X_\gamma\) is the \(d\) given by (12).
Let \( u_\gamma \in F(X_\gamma), \gamma \in \Gamma \). Define \( u \in F(\prod_{\gamma \in \Gamma} X_\gamma) \) as

\[
[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha \text{ for each } \alpha \in (0, 1].
\]  

(13)

We use \( \prod_{\gamma \in \Gamma} u_\gamma \) to denote the fuzzy set \( u \) given by (13).

From Theorem 7.10, \( u \) is well-defined because for each \( \alpha \in (0, 1] \),

\[
[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha = \bigcap_{\beta < \alpha} \prod_{\gamma \in \Gamma} [u_\gamma]_\beta = \bigcap_{\beta < \alpha} [u]_\beta.
\]

In this paper, if not mentioned specially, we use \( \overline{S} \) to denote the closure of \( S \) in a certain extended metric space \((X, d_X)\). For a set \( S \subseteq X_\gamma, \gamma \in \Gamma \), we use \( \overline{S} \) to denote the closure of \( S \) in \((X_\gamma, d_\gamma)\). For a set \( S \subseteq \prod_{\gamma \in \Gamma} X_\gamma \), we also use \( \overline{S} \) to denote the closure of \( S \) in \((\prod_{\gamma \in \Gamma} X_\gamma, d)\). The readers can judge the meaning of \( \overline{S} \) according to the context.

**Lemma 7.11.** Let \( \Gamma \) be a set, and for each \( \gamma \in \Gamma \), let \((X_\gamma, d_\gamma)\) be a metric space. If \( A_\gamma \subseteq X_\gamma \) for \( \gamma \in \Gamma \), then \( \overline{\prod_{\gamma \in \Gamma} A_\gamma} \subseteq \prod_{\gamma \in \Gamma} \overline{A_\gamma} \).

**Proof.** Clearly \( \prod_{\gamma \in \Gamma} A_\gamma \subseteq \prod_{\gamma \in \Gamma} \overline{A_\gamma} \).

Conversely, if \( x = (x_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} \overline{A_\gamma} \), then for each \( \varepsilon > 0 \), there exists \( y = (y_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} A_\gamma \) such that \( d_\gamma(x_\gamma, y_\gamma) \leq \varepsilon \) for all \( \gamma \in \Gamma \). So \( d(x, y) \leq \varepsilon \).

From the arbitrariness of \( \varepsilon > 0 \), we have \( x \in \prod_{\gamma \in \Gamma} A_\gamma \). Thus \( \prod_{\gamma \in \Gamma} A_\gamma \supseteq \prod_{\gamma \in \Gamma} \overline{A_\gamma} \).

In summary, \( \prod_{\gamma \in \Gamma} A_\gamma = \prod_{\gamma \in \Gamma} \overline{A_\gamma} \).

\[ \square \]

**Theorem 7.12.** Let \( \Gamma \) be a set, and for each \( \gamma \in \Gamma \), let \((X_\gamma, d_\gamma)\) be a metric space. If \( u_\gamma \in F_{USC}(X_\gamma) \) for each \( \gamma \in \Gamma \), then \( u = \prod_{\gamma \in \Gamma} u_\gamma \) is a fuzzy set in \( F_{USC}(\prod_{\gamma \in \Gamma} X_\gamma) \).

**Proof.** By (13) and Lemma 7.11, for each \( \alpha \in (0, 1] \),

\[
[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha = [u]_\alpha,
\]

thus \( u \in F_{USC}(\prod_{\gamma \in \Gamma} X_\gamma) \).

\[ \square \]
In the following theorem, we use $H$ to denote the Hausdorff metric on $C(X_\gamma)$ induced by $d_\gamma$. We also use $H$ to denote the Hausdorff metric on $C(\prod_{\gamma \in \Gamma} X_\gamma)$ induced by $d$.

**Theorem 7.13.** Let $\Gamma$ be a set, and for each $\gamma \in \Gamma$, let $(X_\gamma, d_\gamma)$ be a metric space. If $A_\gamma$ and $B_\gamma$ are elements in $C(X_\gamma)$ for $\gamma \in \Gamma$, then $H(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{\gamma \in \Gamma} H(A_\gamma, B_\gamma)$.

**Proof.** From Lemma 7.11, $\prod_{\gamma \in \Gamma} A_\gamma$ and $\prod_{\gamma \in \Gamma} B_\gamma$ are elements in $C(\prod_{\gamma \in \Gamma} X_\gamma)$. Note that $d(x, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{\gamma \in \Gamma} d_\gamma(x_\gamma, B_\gamma)$ for each $x = (x_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} X_\gamma$. Thus

$$H^*(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{x \in \prod_{\gamma \in \Gamma} A_\gamma} d(x, \prod_{\gamma \in \Gamma} B_\gamma)$$

$$= \sup_{x \in \prod_{\gamma \in \Gamma} A_\gamma} \sup_{\gamma \in \Gamma} d_\gamma(x_\gamma, B_\gamma)$$

$$= \sup_{\gamma \in \Gamma} \sup_{x_\gamma \in A_\gamma} d_\gamma(x_\gamma, B_\gamma)$$

$$= \sup_{\gamma \in \Gamma} H^*(A_\gamma, B_\gamma).$$

So

$$H(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{\gamma \in \Gamma} H(A_\gamma, B_\gamma).$$

$\square$

Now, we give an example to show that there exists a metric space $X$ and $u, v \in F^1_{USC}(X)$ such that $H([u]_\alpha, [v]_\alpha)$ is a non-measurable function of $\alpha$ on $[0, 1]$.

**Example 7.14.** We see $[0, 100] \setminus \{10\}$ as a metric subspace of $\mathbb{R}$. Let $z \in (0, 1]$. Define $u^z \in F^1_{USC}([0, 100] \setminus \{10\})$ as

$$[u^z]_\alpha = \begin{cases} 
\{3\}, & \alpha \in [z, 1], \\
\{3\} \cup (10, 10 + \varepsilon), & \alpha = z(1 - \varepsilon), \ 0 < \varepsilon \leq 1.
\end{cases}$$

Let $z \in (0, 1]$. Define $v^z \in F^1_{USC}([0, 100] \setminus \{10\})$ as

$$[v^z]_\alpha = \begin{cases} 
\{73\}, & \alpha \in (z, 1], \\
[71, 81], & \alpha \in [0, z].
\end{cases}$$
Then for \( z \in (0, 1] \),

\[
H([u^z]_\alpha, [v^z]_\alpha) = \begin{cases}
70, & \alpha \in (z, 1], \\
78, & \alpha = z, \\
71 - \varepsilon, & \alpha = z(1 - \varepsilon), 0 < \varepsilon \leq 1,
\end{cases}
\]

where \( H \) is the Hausdorff metric on \( C([0, 100] \setminus \{10\}) \) induced by the metric on \([0, 100] \setminus \{10\}\).

We see \([0, 9] \) as a metric subspace of \( \mathbb{R} \). Define \( w \in F([0, 9]) \) as \( w(t) = 1 \) for all \( t \in [0, 9] \).

Let \( A \) be a non-measurable set in \((0, 1]\).

Let \( u := \prod_{z \in [0, 1]} u_z \) and let \( v := \prod_{z \in [0, 1]} v_z \), where

\[
\begin{align*}
   u_z &= \begin{cases}
u^z, & z \in A, \\
w, & z \in [0, 1] \setminus A,
\end{cases} \\
v_z &= \begin{cases}v^z, & z \in A, \\
w, & z \in [0, 1] \setminus A.
\end{cases}
\end{align*}
\]

Then by Theorem 7.12, \( u \) and \( v \) are fuzzy sets in \( F_{\mathrm{USC}}(\prod_{z \in [0, 1]} X_z) \), where

\[
X_z = \begin{cases} [0, 100] \setminus \{10\}, & z \in A, \\
[0, 9], & z \in [0, 1] \setminus A.
\end{cases}
\]

Here we mention that \( (\prod_{z \in [0, 1]} X_z, d) \) is a metric space with \( d \) given by (12).

By Theorem 7.13,

\[
\begin{align*}
   H([u]_\alpha, [v]_\alpha) &= \sup_{z \in A} H([u^z]_\alpha, [v^z]_\alpha) \vee \sup_{z \in [0, 1] \setminus A} H([0, 9], [0, 9]) \\
   &= \sup_{z \in A} H([u^z]_\alpha, [v^z]_\alpha) \\
   &\leq \begin{cases} 78, & \alpha \in A, \\
                            71, & \alpha \in [0, 1] \setminus A.
\end{cases}
\end{align*}
\]

So \( \{\alpha \in [0, 1] : H([u]_\alpha, [v]_\alpha) > 73\} = A \), and thus \( H([u]_\alpha, [v]_\alpha) \) is a non-measurable function of \( \alpha \) on \([0, 1]\).

In the sequel, we give some improvements of Propositions 7.1, 7.2 and 7.7, which are the statements on measurability of \( H([u]_\alpha, [v]_\alpha) \) presented in [8]. We first prove Theorem 7.15 which is an improvement of Propositions 7.1 and 7.7. Then we show Theorem 7.17 and use it to improve Theorem 7.15 and Proposition 7.2.

Let \( v \in F_{\mathrm{USC}}(X) \) and let \( 0 \leq \alpha < \beta \leq 1 \). The “variation” \( w_v(\alpha, \beta) \) is defined as \( w_v(\alpha, \beta) := \sup\{H([v]_\xi, [v]_\eta) : \xi, \eta \in (\alpha, \beta)]\} \).
Theorem 7.15. Let \( u \in F^1_{USC}(X) \) and let \( v \in F^1_{USCG}(X) \). Then \( H([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\).

Proof. The proof is divided into three steps.

Step (I) \( H^\ast([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\).

Let \( \xi \in \mathbb{R} \) and let \( n \in \mathbb{N} \). Define

\[
S_\xi := \{ \alpha \in [0, 1] : H^\ast([u]_\alpha, [v]_\alpha) \geq \xi \},
\]

\[
S_{\xi, n} := S_\xi \cap \left( \frac{1}{n}, 1 \right].
\]

To show that \( H^\ast([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\), it suffices to show that for each \( \xi \in \mathbb{R} \) and \( n \in \mathbb{N} \), \( S_{\xi, n} \) is a measurable set.

Since \( v \in F^1_{USCG}(X) \), from Lemma 6.5 in [8] for each \( k = 1, 2, \ldots \), there exist \( \alpha_1^{(k)} < \cdots < \alpha_{l_k}^{(k)} = 1 \) such that \( w_v(\alpha_i^{(k)}, \alpha_{i+1}^{(k)}) \leq \frac{1}{k} \) for all \( i = 1, \ldots, l_k - 1 \).

Let \( T_{k, i} := \{ x : \text{there exists } s \in S_\xi \text{ such that } \alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)} \} \). Put \( T_k := \bigcup_{i=1}^{l_k-1} T_{k, i} \). We affirm that

(i) \( T_k \) is a measurable set,

(ii) \( T_k \supseteq S_{\xi, n} \), and

(iii) \( T_k \subseteq S_{\xi - \frac{1}{k}, n} \).

If \( T_{k, i} \neq \emptyset \), then \( T_{k, i} \) is an interval. Thus (i) is true. (ii) follows from the definition of \( T_k \).

For each \( i = 1, \ldots, l_k - 1 \) and each \( x \in T_{k, i} \), there exists an \( s \in S_\xi \) such that \( \alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)} \), and thus

\[
H^\ast([u]_x, [v]_x) 
\geq H^\ast([u]_s, [v]_x) 
\geq H^\ast([u]_s, [v]_s) - H^\ast([v]_x, [v]_s) 
\geq \xi - \frac{1}{k}.
\]

Hence \( T_k \subseteq S_{\xi - \frac{1}{k}} \). Clearly, \( T_k \subseteq \left( \frac{1}{n}, 1 \right] \). So (iii) is proved.

By affirmations (ii) and (iii), we have

\[
S_{\xi, n} \subseteq \bigcap_{k=1}^{+\infty} T_k \subseteq \bigcap_{k=1}^{+\infty} S_{\xi - \frac{1}{k}, n} = S_{\xi, n}.
\]
From affirmation (i), \( \bigcap_{k=1}^{+\infty} T_k \) is measurable, and thus by (15), \( S_{\xi,n} = \bigcap_{k=1}^{+\infty} T_k \) is measurable.

**Step (II)** \( H^*([v]_\alpha, [u]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\).

The proof of Step (II) is similar to that of Step (I).

Let \( \xi \in \mathbb{R} \) and let \( n \in \mathbb{N} \). Define

\[
S^\xi := \{ \alpha \in [0, 1] : H^*([v]_\alpha, [u]_\alpha) \geq \xi \},
\]

\[
S^{\xi,n} := S^\xi \cap \left( \frac{1}{n}, 1 \right).
\]

To show that \( H^*([v]_\alpha, [u]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\), it suffices to show that for each \( \xi \in \mathbb{R} \) and \( n \in \mathbb{N} \), \( S^{\xi,n} \) is a measurable set.

Let \( T^{k,i} := \{ x : \text{there exists } s \in S^\xi \text{ such that } \alpha^{(k)}_i < s \leq x \leq \alpha^{(k)}_{i+1} \} \). Put \( T^k := \bigcup_{i=1}^{k-1} T^{k,i} \). We affirm that

(i') \( T^k \) is a measurable set,

(ii') \( T^k \supseteq S^{\xi,n} \), and

(iii') \( T^k \subseteq S^{\xi-\frac{1}{k},n} \).

(i') is true because if \( T^{k,i} \neq \emptyset \), then \( T^{k,i} \) is a point or an interval. (ii') follows from the definition of \( T^k \).

For each \( i = 1, \ldots, l_k - 1 \) and each \( x \in T^{k,i} \), there exists an \( s \in S^\xi \) such that \( \alpha^{(k)}_i < s \leq x \leq \alpha^{(k)}_{i+1} \), and thus

\[
H^*([v]_x, [u]_x)
\geq H^*([v]_x, [u]_s)
\geq H^*([v]_s, [u]_s) - H^*([v]_s, [v]_x)
\geq \xi - 1/k.
\]

Hence \( T^k \subseteq S_{\xi-\frac{1}{k}} \). Clearly, \( T^k \subseteq \left( \frac{1}{n}, 1 \right) \). So (iii') is proved.

From affirmations (ii') and (iii'),

\[
S^{\xi,n} \subseteq \bigcap_{k=1}^{+\infty} T^k \subseteq \bigcap_{k=1}^{+\infty} S^{\xi-\frac{1}{k},n} = S^{\xi,n}.
\]  \( (16) \)

So by affirmation (i') and (16), \( S^{\xi,n} = \bigcap_{k=1}^{+\infty} T^k \) is measurable.

**Step (III)** \( H([u]_\alpha, [v]_\alpha) \) is a measurable function of \( \alpha \) on \([0, 1]\).
Since that \( H([u]_\alpha,[v]_\alpha) = \max\{H^*([u]_\alpha,[v]_\alpha),H^*([v]_\alpha,[u]_\alpha)\} \), then the desired result follows immediately from the fact that both \( H^*([u]_\alpha,[v]_\alpha) \) and \( H^*([v]_\alpha,[u]_\alpha) \) are measurable functions of \( \alpha \) on \([0,1]\), which is proved in steps (I) and (II).

\[ \square \]

**Remark 7.16.** Theorem 7.15 is an improvement of Proposition 7.7. Since a singleton set is a compact set, Theorem 7.15 is also an improvement of Proposition 7.1.

Obviously, if \( \xi \leq 0 \), then \( S_\xi = S^{\xi} = [0,1] \) and \( S_{\xi,n} = S^{\xi,n} = [\frac{1}{n},1] \).

Let \((X,d_X)\) be a metric subspace of \((Y,d_Y)\). To distinguish from the closure of \( S \) in \((X,d_X)\), we use \( S^Y \) to denote the closure of \( S \) in \((Y,d_Y)\).

For \( u \in F^1_{USC}(X) \), define \( u^Y \in F^1_{USC}(Y) \) as

\[
[u^Y]_\alpha = \cap_{\beta < \alpha} [u]_\beta^Y \quad \text{for} \quad \alpha \in (0,1].
\]

Note that \( [u^Y]_\alpha = \cap_{\beta < \alpha} [u]_\beta^Y \) for all \( \alpha \in (0,1] \) (since we can see that for \( \alpha \in (0,1], \cap_{\beta < \alpha} [u]_\beta^Y = \cap_{\gamma > \beta} [u]_\gamma^Y = \cap_{\gamma > \alpha} [u]_\gamma^Y = [u^Y]_\alpha \). Then by Theorem 7.10, \( u^Y \) is well-defined.

For each \( u \in F^1_{USC}(X) \), define

\[
\Gamma(u)^Y := \{ \alpha \in (0,1] : [u^Y]_\alpha \supseteq [u]_\alpha^Y \}.
\]

If there is no confusion, we will write \( \Gamma(u)^Y \) as \( \Gamma(u) \) for simplicity.

We use \( H \) to denote the Hausdorff metric on \( C(X) \) induced by \( d_X \), and we also use \( H \) to denote the Hausdorff metric on \( C(Y) \) induced by \( d_Y \).

We will use the following Theorem 7.17 to improve Theorem 7.15 and Proposition 7.2.

**Theorem 7.17.** Let \((X,d_X)\) be a metric subspace of \((Y,d_Y)\) and let \( u,v \in F^1_{USC}(X) \). Then

(i) \( [u^Y]_\alpha \supseteq [u]_\alpha^Y \) for all \( \alpha \in (0,1] \), and \( [u^Y]_0 = [u]_0^Y \).

(ii) For each \( \alpha \in [0,1] \setminus (\Gamma(u) \cup \Gamma(v)) \),

\[
H([u^Y]_\alpha,[v^Y]_\alpha) = H([u]_\alpha,[v]_\alpha).
\]

(iii) The cardinality of \( \Gamma(u) \) is less than the cardinality of \( Y \setminus X \).
\textbf{Proof.} (i) follows from the definition of $u^Y$. The proof is routine. From (17), clearly $[u^Y]_\alpha \supseteq [u]_\alpha^Y$. To show $[u^Y]_0 = [u]_0^Y$, it suffices to show that $[u^Y]_0 \supseteq [u]_0^Y$ and $[u^Y]_0 \subseteq [u]_0^Y$. Since $[u^Y]_0 = \bigcup_{\alpha>0} [u^Y]_\alpha \supseteq \bigcup_{\alpha>0} [u]_\alpha^Y = [u]_0^Y$. For each $\alpha \in (0,1]$, $[u^Y]_\alpha \subseteq [u]_{\alpha/2}^Y \subseteq [u]_0^Y$, and hence $[u^Y]_0 = \bigcup_{\alpha>0} [u^Y]_\alpha \subseteq [u]_0^Y$.

From (i) and the definition of $\Gamma(u)$, we have that $\Gamma(u)$ is a measurable function of $\alpha$. Then from Proposition 7.2 and Corollary 7.18, $h$ is an injection. So (iii) is proved.

To show that (iii) is true, it suffices to construct an injection $j : \Gamma(u) \to Y \setminus X$. Let $\gamma \in \Gamma(u)$. Then there is an $x_\gamma \in Y$ such that $x_\gamma \in [u^Y]_\gamma \setminus [\overline{u}^Y]_\gamma$. Define $j(\gamma) = x_\gamma$ for each $\gamma \in \Gamma(u)$. Since $x_\gamma \notin [\overline{u}]_\gamma = \cap_{\beta<\gamma} [\overline{u}]_\beta$, there is a $\beta < \gamma$ such that $x_\beta \notin [\overline{u}]_\beta$. On the other hand, since $x_\gamma \in [u^Y]_\gamma$, we have $x_\gamma \in [\overline{u}]_\beta$. Thus $x_\gamma \in Y \setminus X$. Hence $j$ is a function from $\Gamma(u)$ to $Y \setminus X$.

Let $\xi, \eta \in \Gamma(u)$ with $\xi < \eta$. Since $x_\xi \notin [\overline{u}]_\xi$, then $x_\xi \notin [u^Y]_\lambda$ when $\lambda > \xi$. Hence $x_\xi \notin [u^Y]_\eta$. Notice that $x_\eta \in [u^Y]_\eta$, and therefore $x_\xi \neq x_\eta$. Thus $j$ is an injection. So (iii) is proved. \hfill $\square$

\textbf{Corollary 7.18.} Let $(X, d_X)$ be a metric subspace of $(Y, d_Y)$ and $Y \setminus X$ an at most countable set. Then for $u, v \in F^1_{USC}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0,1]$ if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0,1]$.

\textbf{Proof.} By (ii), (iii) of Theorem 7.17, we have that $H([u]_\alpha, [v]_\alpha) = H([u]_\alpha, [v]_\alpha)$ on $[0,1]$ except at most countable $\alpha \in [0,1]$. Thus we obtain the desired result. \hfill $\square$

Let $S \subseteq \mathbb{R}^m$. We see $\mathbb{R}^m \setminus S$ as a metric subspace of $\mathbb{R}^m$.

\textbf{Corollary 7.19.} Let $S$ be an at most countable subset of $\mathbb{R}^m$. Let $u, v \in F^1_{USC}(\mathbb{R}^m \setminus S)$. Then $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0,1]$.

\textbf{Proof.} The desired result follows from Proposition 7.2 and Corollary 7.18. Let $Y = \mathbb{R}^m$. Then from Proposition 7.2, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0,1]$. So by Corollary 7.18, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of $\alpha$ on $[0,1]$. \hfill $\square$
Corollary 7.20. Let \((X, d_X)\) be a metric subspace of \((Y, d_Y)\) and \(Y \setminus X\) an at most countable set. Let \(u, v \in F^1_{USC}(X)\). If \(u^Y, v^Y \in F^1_{USCG}(Y)\), then 
\[ H([u]_\alpha, [v]_\alpha) \]
is a measurable function of \(\alpha\) on \([0, 1]\).

Proof. The desired result follows from Theorem 7.15 and Corollary 7.18. Since \(v^Y \in F^1_{USC}(Y)\) and \(u^Y \in F^1_{USCG}(Y)\), then by Theorem 7.15, \(H([u^Y]_\alpha, [v^Y]_\alpha)\) is a measurable function of \(\alpha\) on \([0, 1]\). Thus from Corollary 7.18, \(H([u]_\alpha, [v]_\alpha)\) is a measurable function of \(\alpha\) on \([0, 1]\).

\[ \square \]

Remark 7.21. Let \((X, d_X)\) be a metric subspace of \((Y, d_Y)\). Clearly, if \(u \in F^1_{USCG}(X)\), then \([u]_\alpha = [u^Y]_\alpha\) for \(\alpha \in (0, 1]\) and thus \(u^Y \in F^1_{USCG}(Y)\). So Corollary 7.20 is an improvement of Theorem 7.15.

Corollary 7.19 is an improvement of Proposition 7.2. Theorem 7.15 is the special case of Corollary 7.20 when \(Y = X\). Proposition 7.2 is the special case of Corollary 7.19 when \(S = \emptyset\).

The results in this section was recorded in [12]. In essence, contents including Theorem 7.17, Corollaries 7.18 and 7.19 have already been proved in chinaxiv:202108.00116v1, which is a previous version of [12].

8. Conclusions

In this paper, we discuss properties of the \(d_p\) metrics and the spaces of fuzzy sets in a general metric space \((X, d)\) with \(d_p\) metrics.

We show that for \(u, v \in F^1_{USC}(X)\), \(d^*_p(u, v) \geq \left(\frac{H_{\text{end}}([u]_\alpha, [v]_\alpha)}{p+1}\right)^{1/p}\). If \(d_p(u, v)\) is well-defined, then \(d_p(u, v) = d^*_p(u, v)\) and of course \(d^*_p(u, v)\) can be replaced by \(d^*_p(u, v)\) in the above inequality.

We obtain the characterizations of total boundedness, relative compactness and compactness in \((F^1_{USCG}(X)^p, d_p)\). These conclusions generalize the corresponding conclusions in [10]. Our results indicate that for a subset \(U\) in \(F^1_{USCG}(X)^p\), \(U\) is total bounded (respectively, relatively compact, compact) in \((F^1_{USCG}(X)^p, d_p)\) if and only if \(U\) is total bounded (respectively, relatively compact, compact) in \((F^1_{USCG}(X), H_{\text{end}})\) and \(U\) is \(p\)-mean equi-continuous.

We show that \((F^1_{USCG}(\bar{X})^p, d_p)\) is a completion of \((F^1_{USCB}(X), d_p)\), and thus a completion of \((F^1_{USCG}(X)^p, d_p)\).

In what cases the \(d_p\) metrics are well-defined is a fundamental question. For \(u, v \in F^1_{USC}(X)\), \(d_p(u, v)\) is well-defined if and only if \(H([u]_\alpha, [v]_\alpha)\) is a
measurable function of $\alpha$ on $[0, 1]$. In Section 7 of this paper, we obtain the following conclusion.

(i) Let $(X, d_X)$ be a metric subspace of $(Y, d_Y)$ and $Y \setminus X$ an at most countable set. Let $u, v \in F^1_{USC}(X)$. If $u^Y \in F^1_{USCG}(Y)$, then $d_p(u, v)$ is well-defined.

In the special case of $Y = X$, the above conclusion become:

Let $u \in F^1_{USC}(X)$ and let $v \in F^1_{USCG}(X)$. Then $d_p(u, v)$ is well-defined.

(ii) Let $S$ be an at most countable subset of $\mathbb{R}^m$. We see $\mathbb{R}^m \setminus S$ as a metric subspace of $\mathbb{R}^m$. For $u, v \in F^1_{USC}(\mathbb{R}^m \setminus S)$, $d_p(u, v)$ is well-defined.

In the special case of $S = \emptyset$, the above conclusion become:

For $u, v \in F^1_{USC}(\mathbb{R}^m)$, $d_p(u, v)$ is well-defined (we point out this conclusion in [8]).

(iii) There exists a metric space $X$ and $u, v \in F^1_{USC}(X)$ such that $d_p(u, v)$ is not well-defined (we point out this conclusion in [8]). In [8], we introduce the $d^*_p$ metric on $F^1_{USC}(X)$, which is an expansion of the $d_p$ distance on $F^1_{USC}(X)$.

The results of this paper have potential applications in the work relevant to $d_p$ distance on fuzzy sets.


[11] H. Huang, Measurability of functions induced from fuzzy sets, submitted to the preprint of National Science and Technology Library on 2021.05.28


