Characterizations of compactness in fuzzy set spaces

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Abstract
In this paper, we discuss characterizations of total boundedness, relative compactness and compactness in \((F_{USCG}(X), H_{end})\) and \((F_{USCB}(X), H_{end})\), respectively. We also give completions of \((F_{USCG}(X), H_{end})\) and \((F_{USCB}(X), H_{end})\), respectively.

Keywords: Endograph metric; Hausdorff metric; Total boundedness; Relative compactness; Compactness

1. Fuzzy sets and convergence structures on them
Let \((X, d)\) be a metric space and let \(K(X)\) and \(C(X)\) denote the set of all non-empty compact subsets of \(X\) and the set of all non-empty closed subsets of \(X\), respectively.

We use \(H\) to denote the Hausdorff metric on \(C(X)\) induced by \(d\), i.e.,

\[
H(U, V) = \max\{H^*(U, V), H^*(V, U)\}
\]

for arbitrary \(U, V \in C(X)\), where

\[
H^*(U, V) = \sup_{u \in U} d(u, V) = \sup_{u \in U} \inf_{v \in V} d(u, v).
\]

The metric \(\bar{d}\) on \(X \times [0, 1]\) is defined as

\[
\bar{d}(x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.
\]

If there is no confusion, we also use \(H\) to denote the Hausdorff metric on \(C(X \times [0, 1])\) induced by \(\bar{d}\).

The Hausdorff metric has the following important properties.

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Theorem 1.1. [1] Let $(X,d)$ be a metric space and let $H$ be the Hausdorff metric induced by $d$. Then the following statements are true.

(i) $(X,d)$ is complete $\iff$ $(K(X),H)$ is complete

(ii) $(X,d)$ is separable $\iff$ $(K(X),H)$ is separable

(iii) $(X,d)$ is compact $\iff$ $(K(X),H)$ is compact

Let $(X,d)$ be a metric space. We say that a sequence of sets $\{C_n\}$ Kuratowski converges to $C \subseteq X$, if

$$C = \liminf_{n \to \infty} C_n = \limsup_{n \to \infty} C_n,$$

where

$$\liminf_{n \to \infty} C_n = \{ x \in X : x = \lim_{n \to \infty} x_n, x_n \in C_n \},$$

$$\limsup_{n \to \infty} C_n = \{ x \in X : x = \lim_{j \to \infty} x_{n_j}, x_{n_j} \in C_{n_j} \} = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} C_m.$$

In this case, we’ll write $C = \lim_{n \to \infty} C_n(Kuratowski)$ or $C = \lim_{n \to \infty} C_n(K)$ for simplicity.

Let $F(X)$ denote the set of all fuzzy sets in $X$. A fuzzy set $u \in F(X)$ can be seen as a function $u : X \to [0,1]$. In this sense, a subset $S$ of $X$ can be seen as a fuzzy set $S_{F(X)}$ in $X$

$$S_{F(X)}(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For $x \in X$, we use $\hat{x}_X$ to denote the fuzzy set $\{x\}_{F(X)}$ in $X$. If there is no confusion, we will write $\hat{x}_X$ as $\hat{x}$ for simplicity.

For $u \in F(X)$, let $[u]_\alpha$ denote the $\alpha$-cut of $u$, i.e.

$$[u]_\alpha = \begin{cases} \{ x \in X : u(x) \geq \alpha \}, & \alpha \in (0,1], \\ \text{supp } u = \{ u > 0 \}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the closure of $S$ in $(X,d)$.

For $u \in F(X)$, define

$$\text{end } u := \{(x,t) \in X \times [0,1] : u(x) \geq t\},$$

$$\text{send } u := \{(x,t) \in X \times [0,1] : u(x) \geq t\} \cap ([u]_0 \times [0,1]).$$
end \( u \) and send \( u \) are called the endograph and the sendograph of \( u \), respectively.

Let \( F_{USC}(X) \) denote the set of all normal and upper semi-continuous fuzzy sets \( u : X \to [0,1] \), i.e.,

\[
F_{USC}(X) := \{ u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0,1] \}.
\]

Rojas-Medar and Román-Flores [2] have introduced the \( \Gamma \)-convergence on \( F_{USC}(X) \):

Let \( u, u_n, n = 1, 2, \ldots \), be fuzzy sets in \( F_{USC}(X) \). Then \( u_n \ \Gamma \)-converges to \( u \) if

\[
\text{end } u = \lim_{n \to \infty} \text{end } u_n (K).
\]

The endograph metric \( H_{end} \) and the sendograph metric \( H_{send} \) can be defined on \( F_{USC}(X) \) as usual. For \( u, v \in F_{USC}(X) \),

\[
H_{end}(u, v) := H(\text{end } u, \text{end } v),
\]

\[
H_{send}(u, v) := H(\text{send } u, \text{send } v).
\]

The endograph metric \( H_{end} \) and the sendograph metric \( H_{send} \) are defined by using the Hausdorff metric on \( C(X \times [0,1]) \) induced by \( \bar{d} \) on \( X \times [0,1] \).

**Definition 1.2.** Define two subsets of \( F_{USC}(X) \) as follows.

\[
F_{USCB}(X) := \{ u \in F_{USC}(X) : [u]_0 \in K(X) \},
\]

\[
F_{USCG}(X) := \{ u \in F_{USC}(X) : [u]_\alpha \in K(X) \text{ for all } \alpha \in (0,1] \}.
\]

Readers can refer to [4] for more contents.

2. Level characterizations of \( \Gamma \)-convergence

In this section, we investigate the level characterizations of the \( \Gamma \)-convergence. It is found that the \( \Gamma \)-convergence has the level decomposition property on \( F_{USCG}(X) \), fuzzy sets in which has compact positive \( \alpha \)-cuts. It is pointed out that the \( \Gamma \)-convergence need not have the level decomposition property on \( F_{USC}(X) \).

Rojas-Medar and Román-Flores [2] have introduced the following useful property of \( \Gamma \)-convergence.
Theorem 2.1. [2] Suppose that $u, u_n, n = 1, 2, \ldots,$ are fuzzy sets in $F_{USC}(X)$. Then $u_n \xrightarrow{\Gamma} u$ iff for all $\alpha \in (0, 1],$

$$\{u > \alpha\} \subseteq \liminf_{n \to \infty} [u_n]_\alpha \subseteq \limsup_{n \to \infty} [u_n]_\alpha \subseteq [u]_\alpha. \quad (1)$$

Remark 2.2. Rojas-Medar and Román-Flores (Proposition 3.5 in [2]) presented the statement in Proposition 2.1 when $u, u_n, n = 1, 2, \ldots,$ are fuzzy sets in $E^m$. It can be checked that this conclusion also holds when $u, u_n, n = 1, 2, \ldots,$ are fuzzy sets in $F_{USC}(X)$.

Theorem 2.3. [6] Let $(X, d)$ be a metric space and let $\{C_n\}$ be a sequence of sets in $X$. Then $\liminf_{n \to \infty} C_n$ and $\limsup_{n \to \infty} C_n$ are closed sets.

Theorem 2.4. Suppose that $u, u_n, n = 1, 2, \ldots,$ are fuzzy sets in $F_{USC}(X)$. Then $u_n \xrightarrow{\Gamma} u$ iff for all $\alpha \in (0, 1],$

$$\{u > \alpha\} \subseteq \liminf_{n \to \infty} [u_n]_\alpha \subseteq \limsup_{n \to \infty} [u_n]_\alpha \subseteq [u]_\alpha.$$

Lemma 2.5. Let $U_n \in K(X)$ for $n = 1, 2, \ldots$.

(i) If $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$, then $\bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \to 0$ as $n \to +\infty$.

(ii) If $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq \ldots$ and $U = \bigcup_{n=1}^{+\infty} U_n \in K(X)$, then $H(U_n, U) \to 0$ as $n \to +\infty$.

Proof. This is Lemma 4.4 in [3].

Let $u$ be a fuzzy set in $F_{USC}(X)$. Denote

- $P(u) := \{\alpha \in (0, 1) : \{u > \alpha\} \subsetneq\not\subset [u]_\alpha\}$.
- $P_0(u) := \{\alpha \in (0, 1) : \lim_{\beta \to \alpha} H([u]_\beta, [u]_\alpha) \neq 0\}$.

A number $\alpha$ in $P(u)$ is called a platform point of $u$. Clearly, $P(u) \subseteq P_0(u)$. From Lemma 2.5, we can obtain that $P(u) = P_0(u)$ for $u \in F_{USCG}(X)$.

Lemma 2.6. Given $u \in F_{USCG}(X)$. Then $P_0(u) = P(u)$ and $P(u)$ is at most countable.

Theorem 2.7. Suppose that $u, u_n, n = 1, 2, \ldots,$ are fuzzy sets in $F_{USC}(X)$. Then the following statements are true.
(i) If \([u]_\alpha = \lim_{n \to \infty} [u_n]_\alpha (K)\) for \(\alpha \in P\), where \(P\) is a dense set in \((0, 1)\), then \(u_n \Gamma \to u\).

(ii) If \(u_n \Gamma \to u\), then \([u]_\alpha = \lim_{n \to \infty} [u_n]_\alpha (K)\) for all \(\alpha \in (0, 1) \setminus P(u)\).

The following theorem states the level decomposition property of \(\Gamma\)-convergence on compact positive \(\alpha\)-cuts fuzzy sets in \(F_{USC}(X)\).

**Theorem 2.8.** Suppose that \(u, u_n, n = 1, 2, \ldots\), are fuzzy sets in \(F_{USC}(X)\). Then the following statements are true.

(i) \(u_n \Gamma \to u\)

(ii) \(u_n \text{ a.e.} \to u(K)\).

(iii) \([u]_\alpha = \lim_{n \to \infty} [u_n]_\alpha (K)\) for all \(\alpha \in (0, 1) \setminus P(u)\)

(iv) \(\lim_{n \to \infty} [u_n]_\alpha (K) = [u]_\alpha\) holds when \(\alpha \in P\), where \(P\) is a dense subset of \((0, 1) \setminus P(u)\).

(v) \(\lim_{n \to \infty} [u_n]_\alpha (K) = [u]_\alpha\) holds when \(\alpha \in P\), where \(P\) is a countable dense subset of \((0, 1) \setminus P(u)\).

**Remark 2.9.** It can be checked that the level decomposition property of \(\Gamma\)-convergence need not hold on \(F_{USC}(X)\).

3. Level characterizations of endograph metric convergence

In this section, we discuss the level characterizations of endograph metric convergence.

The following are two elementary conclusions which are useful in this paper.

**Theorem 3.1.** Suppose that \(C, C_n\) are sets in \(C(X)\), \(n = 1, 2, \ldots\). Then \(H(C_n, C) \to 0\) implies that \(\lim_{n \to \infty} C_n (K) = C\).

**Proof.** This is an already known result. Its proof is similar to that of Theorem 4.1 in [6].

**Proposition 3.2.** Given \(u, u_n, n = 1, 2, \ldots\), in \(F_{USC}(X)\). Then
(i) $H_{\text{end}}(u_n, u) \to 0$ is equivalent to $H_{\text{end}}(u_n, u) \to 0$ and $H([u_n]_0, [u]_0) \to 0$

(ii) $\lim_{n \to \infty} \text{send } u_n = \text{send } u$ is equivalent to $u_n \xrightarrow{\Gamma} u$ and $\lim_{n \to \infty} [u_n]_0(K) = [u]_0$

**Theorem 3.3.** Let $u, u_n, n = 1, 2, \ldots$, be fuzzy sets in $F_{\text{USC}}(X)$ and let $P$ be a dense subset of $[0, 1]$. Suppose that $H([u_n]_\alpha, [u]_\alpha) \to 0$ for each $\alpha \in P$. Then $H_{\text{end}}(u_n, u) \to 0$.

**Proof.** We proceed by contradiction. If $H_{\text{end}}(u_n, u) \not\to 0$, then there is an $\varepsilon > 0$ such that $H_{\text{end}}(u_n, u) > \varepsilon$ for a subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Suppose $H^*(\text{end } u_n, \text{end } u) > \varepsilon$. Then there exists a sequence $(x_{n_k}, \alpha_{n_k}) \in \text{end } u_{n_k}$ such that

$$d((x_{n_k}, \alpha_{n_k}), \text{end } u) > \varepsilon. \quad (2)$$

With no loss of generality we can assume $\alpha_{n_k} \to \alpha \geq \varepsilon$. Pick $\beta \in P$ satisfying $\alpha \in (\beta, \beta + \varepsilon/2)$. Then there exists $K$ such that $\alpha_{n_k} \in (\beta, \beta + \varepsilon/2)$ for all $k \geq K$. Thus for each $k \geq K$,

$$d((x_{n_k}, \alpha_{n_k}), \text{end } u) \leq d((x_{n_k}, \beta), \text{end } u) + \varepsilon/2 \leq H([u_{n_k}]_\beta, [u]_\beta) + \varepsilon/2 \quad (3)$$

Note that $H([u_{n_k}]_\beta, [u]_\beta) \to 0$, thus (2) contradicts (3). So the supposition is false.

Suppose $H^*(\text{end } u, \text{end } u_{n_k}) > \varepsilon$. Then similarly we can derive a contradiction.

\[ \square \]

**Remark 3.4.** Fan (Lemma 1 in [5]) proved a result of Theorem 3.3 type.

**Theorem 3.5.** Let $u, u_n, n = 1, 2, \ldots$, be fuzzy sets in $F_{\text{USC}}(X)$. Suppose that $H_{\text{end}}(u_n, u) \to 0$. Then $H([u_n]_\alpha, [u]_\alpha) \to 0$ for each $\alpha \in (0, 1) \setminus P_0(u)$

**Proof.** Let $\alpha \in (0, 1) \setminus P_0(u)$. Given $\varepsilon > 0$. Then there exists a $\delta(\alpha, \varepsilon) \in (0, \varepsilon/2)$ such that $[\alpha - \delta, \alpha + \delta] \subset [0, 1]$ and

$$H([u]_\beta, [u]_\alpha) < \varepsilon/2 \quad (4)$$

for all $\beta \in [\alpha - \delta, \alpha + \delta]$.
From $H_{\text{end}}(u_n, u) \to 0$, there exists an $N(\delta)$ such that
\[ H_{\text{end}}(u_n, u) < \delta \]  \quad (5)
for all $n \geq N$. Thus
\[ H^*([u_n]_\alpha, [u]_\alpha - \delta) < \delta < \varepsilon/2. \]

So, for each $n \geq N$,
\begin{align*}
H^*([u_n]_\alpha, [u]_\alpha) & \leq H^*([u_n]_\alpha, [u]_\alpha - \delta) + H([u]_\alpha, [u]_\alpha - \delta) \\
& < \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{6}
\end{align*}

Similarly, it follows from (5) that
\[ H^*([u]_\alpha + \delta, [u_n]_\alpha) < \delta < \varepsilon/2, \]
and then, for each $n \geq N$,
\begin{align*}
H^*([u]_\alpha + \delta, [u_n]_\alpha) & \leq H([u]_\alpha, [u]_\alpha + \delta) + H^*([u]_\alpha + \delta, [u_n]_\alpha) \\
& < \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{7}
\end{align*}

Combined with (6) and (7),
\[ H([u]_\alpha, [u_n]_\alpha) \to 0. \]

The following theorem gives the level decomposition property of $H_{\text{end}}$ on $F_{USCG}(X)$.

**Theorem 3.6.** Suppose that $u, u_n, n = 1, 2, \ldots$, are fuzzy sets in $F_{USCG}(X)$. Then the following statements are equivalent.

(i) $H_{\text{end}}(u_n, u) \to 0$

(ii) $H([u_n]_\alpha, [u]_\alpha) \xrightarrow{a.e.} 0$

(iii) $H([u_n]_\alpha, [u]_\alpha) \to 0$ for all $\alpha \in (0, 1) \setminus P(u)$
(iv) $H([u_n]_\alpha, [u]_\alpha) \to 0$ when $\alpha \in P$, where $P$ is a dense subset of $(0,1) \setminus P(u)$

(v) $H([u_n]_\alpha, [u]_\alpha) \to 0$ when $\alpha \in P$, where $P$ is a countable dense subset of $(0,1) \setminus P(u)$

**Proof.** The desired result follows from Lemma 2.6 and Theorems 3.3 and 3.5. \qed

**Remark 3.7.** It can be checked that the level decomposition property of $H_{\text{end}}$ convergence need not hold on $F_{\text{USC}}(X)$.

4. Characterizations of compactness in $(F_{\text{USCG}}(X), H_{\text{end}})$

Based on the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F_{\text{USCG}}(X), H_{\text{end}})$. We use $(\tilde{\mathcal{X}}, \tilde{d})$ to denote the completion of $(\mathcal{X}, d)$. We see $(\mathcal{X}, d)$ as a subspace of $(\tilde{\mathcal{X}}, \tilde{d})$. Let $S \subseteq \tilde{\mathcal{X}}$. The symbol $\tilde{S}$ is used to denote the closure of $S$ in $(\tilde{\mathcal{X}}, \tilde{d})$.

As defined in Section 1, we have $K(\tilde{\mathcal{X}}), C(\tilde{\mathcal{X}}), F_{\text{USC}}(\tilde{\mathcal{X}}), F_{\text{USCG}}(\tilde{\mathcal{X}})$, etc. according to $(\tilde{\mathcal{X}}, \tilde{d})$. For example,

$$F_{\text{USC}}(\tilde{\mathcal{X}}) := \{ u \in F(\tilde{\mathcal{X}}) : [u]_\alpha \in C(\tilde{\mathcal{X}}) \text{ for all } \alpha \in [0,1] \},$$

$$F_{\text{USCG}}(\tilde{\mathcal{X}}) := \{ u \in F(\tilde{\mathcal{X}}) : [u]_\alpha \in K(\tilde{\mathcal{X}}) \text{ for all } \alpha \in (0,1] \}.$$

If there is no confusion, we also use $H$ to denote the Hausdorff metric on $C(\tilde{\mathcal{X}})$ induced by $\tilde{d}$. We also use $H$ to denote the Hausdorff metric on $C(\tilde{\mathcal{X}} \times [0,1])$ induced by $\tilde{d}$. We also use $H_{\text{end}}$ to denote the endograph metric on $F_{\text{USC}}(\tilde{\mathcal{X}})$ given by using $H$ on $C(\tilde{\mathcal{X}} \times [0,1])$.

Clearly, the induced metric on $F_{\text{USCG}}(X)$ by the $H_{\text{end}}$ on $F_{\text{USC}}(X)$ is the same as the induced metric on $F_{\text{USCG}}(X)$ by the $H_{\text{end}}$ on $F_{\text{USC}}(\tilde{\mathcal{X}})$.

We see $(F_{\text{USCG}}(X), H_{\text{end}})$ as a subspace of $(F_{\text{USCG}}(\tilde{\mathcal{X}}), H_{\text{end}})$.

4.1. Characterizations of compactness in $(K(X), H)$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(K(X), H)$. The results in this subsection are basis for contents in the sequel.
**Theorem 4.1.** Suppose that \((X,d)\) is complete and that \(\{C_n\}\) is a Cauchy sequence in \((K(X),H)\). Let \(D_n = \bigcup_{l=1}^{n} C_l\) and \(D = \bigcup_{l=1}^{+\infty} C_l\). Then \(D \in K(X)\) and \(H(D_n, D) \to 0\).

**Proof.** Note that for \(k > j\),
\[
H(D_k, D_j) \leq \max\{H(C_i, C_j) : i = j + 1, \ldots, k\}.
\]
So \(\{D_n\}\) is a Cauchy sequence in \((K(X),H)\). From Theorem 1.1, \((K(X),H)\) is complete, and thus \(\{D_n\}\) converges to \(D \in K(X)\). \qed

**Theorem 4.2.** Suppose that \((X,d)\) is a metric space and that \(\mathcal{D} \subseteq K(X)\). Then \(\mathcal{D}\) is totally bounded in \((K(X),H)\) is equivalent to \(\mathcal{D} = \bigcup \{C : C \in \mathcal{D}\}\) is totally bounded in \((X,d)\).

**Proof.** If \(\mathcal{D} = \emptyset\), then the desired result follows immediately. Suppose that \(\mathcal{D} \neq \emptyset\).

**Necessity.** To show that \(\mathcal{D}\) is totally bounded. We only need to show that each sequence in \(\mathcal{D}\) has a Cauchy subsequence.

Given a sequence \(\{x_n\}\) in \(\mathcal{D}\). Suppose that \(x_n \in C_n \in \mathcal{D}\). Since \(\mathcal{D}\) is totally bounded, then \(\{C_n\}\) has a Cauchy subsequence \(\{C_{n_k}\}\). Hence, by Theorem 4.1, \(\bigcup_{k=1}^{+\infty} C_{n_k}\) is in \(K(\tilde{X})\). Thus \(\{x_n\}\) has a Cauchy subsequence.

**Sufficiency.** If \(\mathcal{D}\) is totally bounded in \(X\), then \(\tilde{\mathcal{D}}\) is in \(K(\tilde{X})\). So, by Theorem 1.1, \((K(\tilde{\mathcal{D}}), H)\) is compact, and thus \(\mathcal{D}\) is totally bounded. \qed

**Theorem 4.3.** Suppose that \((X,d)\) is a metric space and that \(\mathcal{D} \subseteq K(X)\). Then \(\mathcal{D}\) is relatively compact in \((K(X),H)\) is equivalent to \(\mathcal{D} = \bigcup \{C : C \in \mathcal{D}\}\) is relatively compact in \((X,d)\).

**Proof.** If \(\mathcal{D} = \emptyset\), then the desired result follows immediately. Suppose that \(\mathcal{D} \neq \emptyset\).

**Necessity.** To show that \(\mathcal{D}\) is relatively compact. We only need to show that each sequence in \(\mathcal{D}\) has a convergent subsequence in \(X\).

Given a sequence \(\{x_n\}\) in \(\mathcal{D}\). Suppose that \(x_n \in C_n \in \mathcal{D}\). Since \(\mathcal{D}\) is relatively compact, then \(\{C_n\}\) has a subsequence \(\{C_{n_k}\}\) converges to \(C\) in \(K(X)\). Hence, by Theorem 4.1, \(\bigcup_{k=1}^{+\infty} C_{n_k}\) is in \(K(\tilde{X})\) (In fact, \(\bigcup_{k=1}^{+\infty} C_{n_k}\) is in
So \( \{x_{n_k}\} \) has a subsequence which converges to \( x \) in \( \bigcup_{k=1}^{+\infty} C_{n_k} \), and thus \( x \in C \subset X \).

**Sufficiency.** If \( D \) is relatively compact in \( X \), then \( D \) is in \( K(X) \), and therefore \((K(D), H)\) is compact. Thus \( D \subset K(D) \) is relatively compact in \((K(X), H)\).

\[ \]

**Lemma 4.4.** Suppose that \((X, d)\) is a metric space and that \( D \subseteq K(X) \). If \( D \) is compact in \((K(X), H)\), then \( D = \bigcup \{ C : C \in D \} \) is compact in \((X, d)\).

**Proof.** If \( D = \emptyset \), then the desired result follows immediately. Suppose that \( D \neq \emptyset \). To show that \( D \) is compact. We only need to show that each sequence in \( D \) has a subsequence converges to a point in \( D \).

Given a sequence \( \{x_n\} \) in \( D \). Suppose that \( x_n \in C_n \in D \). Since \( D \) is compact, then \( \{C_n\} \) has a subsequence \( \{C_{n_k}\} \) converges to \( C \in D \). Hence, by Theorem 4.1, \( \bigcup_{k=1}^{+\infty} C_{n_k} \) is in \( K(\tilde{X}) \) (In fact, \( \bigcup_{k=1}^{+\infty} C_{n_k} \) is in \( K(D) \)). So \( \{x_{n_k}\} \) has a subsequence which converges to \( x \) in \( \bigcup_{k=1}^{+\infty} C_{n_k} \). Thus \( x \in C \subset D \).

**Remark 4.5.** The converse of the implication in Lemma 4.4 does not hold. Let \((X, d) = \mathbb{R} \) and let \( D = \{[0, x] : x \in (0.3, 1]\} \subset K(\mathbb{R}) \). Then \( D = [0, 1] \in K(\mathbb{R}) \). But \( D \) is not compact in \((K(\mathbb{R}), H)\).

**Theorem 4.6.** Suppose that \((X, d)\) is a metric space and that \( D \subseteq K(X) \). Then the following statements are equivalent:
(i) \( D \) is compact in \((K(X), H)\)
(ii) \( D = \bigcup \{ C : C \in D \} \) is relatively compact in \((X, d)\) and \( D \) is closed in \((K(X), H)\)
(iii) \( D = \bigcup \{ C : C \in D \} \) is compact in \((X, d)\) and \( D \) is closed in \((K(X), H)\).

**Proof.** The desired result follows from Theorem 4.3 and Lemma 4.4.

**Remark 4.7.** We are not sure whether Theorems 4.2, 4.3, and 4.6 are already existing results previously, so we give our proofs here. Now we find Theorem 4.3 is Proposition 5 in [8].

### 4.2. Characterizations of compactness in \((F_{USCG}(X), H_{end})\)

Suppose that \( U \) is a subset of \( F_{USC}(X) \) and \( \alpha \in [0, 1] \). For writing convenience, we denote
\begin{itemize}
\item $U(\alpha) := \bigcup_{u \in U} [u]_\alpha$, and
\item $U_\alpha := \{ [u]_\alpha : u \in U \}$.
\end{itemize}

**Theorem 4.8.** Let $U$ be a subset of $F_{USCG}(X)$. Then $U$ is totally bounded in $(F_{USCG}(X), H_{end})$ if and only if $U(\alpha)$ is totally bounded in $(X, d)$ for each $\alpha \in (0, 1]$

**Proof.** **Necessity.** Suppose that $U$ is totally bounded in $(F_{USCG}(X), H_{end})$. To show that $U(\alpha)$ is totally bounded in $X$, we only need to show that each sequence in $U(\alpha)$ has a Cauchy subsequence.

Let $\alpha \in (0, 1]$. Given $\{x_n\} \subset U(\alpha)$. Suppose that $x_n \in [u_n]_\alpha$, $u_n \in U$, $n = 1, 2, \ldots$. Then $\{u_n\}$ has a Cauchy subsequence $\{u_{n_k}\}$. So given $\varepsilon \in (0, \alpha)$, there is a $K(\varepsilon) \in \mathbb{N}$ such that

$$H_{end}(u_{n_l}, u_{n_K}) < \varepsilon$$

for all $l \geq K$. Thus

$$H^*([u_{n_l}]_\alpha, [u_{n_K}]_{\alpha-\varepsilon}) < \varepsilon$$

(8) for all $l \geq K$. From (8) and the arbitrariness of $\varepsilon$, $\bigcup_{k=1}^{+\infty} [u_{n_k}]_\alpha$ is totally bounded in $X$. Thus $\{x_{n_k}\}$ has a Cauchy subsequence, and so does $\{x_n\}$.

**Sufficiency.** Suppose that $U(\alpha)$ is totally bounded in $X$ for each $\alpha \in (0, 1]$. By Theorem 4.2, $U(\alpha)$ is totally bounded in $X$ is equivalent to $U_\alpha$ is totally bounded in $(K(X), H)$. Thus, by Theorem 1.1, we have the following affirmation

- Given $\alpha \in (0, 1]$. For each sequence $\{[u_n]_\alpha, n = 1, 2, \ldots\}$ in $U_\alpha$, it has a subsequence $\{[u_{n_k}]_\alpha, k = 1, 2, \ldots\}$ which converges to $u_\alpha \in K(\bar{X})$ with respect to the Hausdorff metric $H$.

To prove that $U$ is totally bounded, it suffices to show that each sequence in $U$ has a convergent subsequence in $(F_{USCG}(\bar{X}), H_{end})$. Suppose that $\{u_n\}$ is a sequence in $U$. Based on the above affirmation and Theorem 3.6, and proceeding similarly to the proof of the “Sufficiency part” of Theorem 7.1 in [6], it can be shown that $\{u_n\}$ has a subsequence $\{v_n\}$ which converges to $v \in F_{USCG}(\bar{X})$ with respect to $H_{end}$.

A sketch of the proof of the existence of $\{v_n\}$ and $v$ is given as follows.

First, we construct a subsequence $\{v_n\}$ of $\{u_n\}$ such that $[v_n]_q$ converges to $u_q \in K(\bar{X})$ according to the Hausdorff metric $H$ for all $q \in Q'$, where $Q' = Q \cap (0, 1]$. Then we show that $v \in F_{USCG}(\bar{X})$ with $[v]_\alpha = \bigcap_{q < \alpha, \alpha \in Q'} u_q$ for all $\alpha \in (0, 1]$ satisfies that $H_{end}(v_n, v) \to 0$. \qed
Remark 4.9. Some of the implications in the proofs of this paper are actually the equivalent. For example, in the proof of Theorem 4.8, \( U(\alpha) \) is totally bounded in \( X \) for each \( \alpha \in (0,1] \) is equivalent to the affirmation after the “•”

Theorem 4.10. Let \( U \) be a subset of \( F_{USCG}(X) \). Then \( U \) is relatively compact in \( (F_{USCG}(X), H_{end}) \) if and only if \( U(\alpha) \) is relatively compact in \( (X,d) \) for each \( \alpha \in (0,1] \).

Proof. Necessity. Suppose that \( U \) is relatively compact. Given \( \alpha \in (0,1] \). To show that \( U(\alpha) \) is relatively compact in \( X \), we only need to show that each sequence in \( U(\alpha) \) has a convergent subsequence in \( X \).

Let \( \{x_n\} \) be a sequence in \( U(\alpha) \). Suppose that \( x_n \in [u_n]_\alpha \), \( u_n \in U \), \( n = 1,2,\ldots \) Then there is a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) and \( u \in F_{USCG}(X) \) such that \( H_{end}(u_{n_k}, u) \to 0 \). So, by Theorem 3.6, \( H([u_{n_k}]_\alpha, [u]_\alpha) \xrightarrow{\delta} 0 \), and therefore there is a \( \beta \in (0,\alpha) \) such that \( H([u_{n_k}]_\beta, [u]_\beta) \to 0 \). Hence by Theorem 4.3, \( \bigcup_{k=1}^{+\infty} [u_{n_k}]_\beta \) is relatively compact in \( X \). Thus \( \{x_{n_k}\} \) has a convergent subsequence in \( X \), and so does \( \{x_n\} \).

Sufficiency. Suppose that \( U(\alpha) \) is relatively compact in \( X \) for each \( \alpha \in (0,1] \). To show that \( U \) is relatively compact in \( (F_{USCG}(X), H_{end}) \), we only need to show that each sequence in \( U \) has a convergent subsequence in \( (F_{USCG}(X), H_{end}) \).

By Theorem 4.3, \( U(\alpha) \) is relatively compact in \( X \) is equivalent to \( U_\alpha \) is relatively compact in \( K(X) \). Thus, we have the following affirmation

- Given \( \alpha \in (0,1] \). For each sequence \( \{[u_n]_\alpha, n = 1,2,\ldots\} \) in \( U_\alpha \), it has a subsequence \( \{[u_{n_k}]_\alpha, k = 1,2,\ldots\} \) which converges to \( u_\alpha \in K(X) \) with respect to the Hausdorff metric \( H \).

The remaining proof is similar to the corresponding part of the “Sufficiency part” of Theorem 4.8.

We can also prove that \( U \) is relatively compact in \( (F_{USCG}(X), H_{end}) \) as follows. From the “Sufficiency part” of Theorem 4.8, we know that for each sequence \( \{u_n\} \) in \( U \), there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) which converges to \( v \in F_{USCG}(\tilde{X}) \). From Theorem 3.6 and the above statement after the “•”, we thus know that \( v \in F_{USCG}(X) \).

\[ \square \]

Theorem 4.11. Let \( U \) be a subset of \( F_{USCG}(X) \). Then the following statements are equivalent.
(i) $U$ is compact in $(F_{USCG}(X), H_{end})$

(ii) $U(\alpha)$ is relatively compact in $(X, d)$ for each $\alpha \in (0, 1]$ and $U$ is closed in $(F_{USCG}(X), H_{end})$

(iii) $U(\alpha)$ is compact in $(X, d)$ for each $\alpha \in (0, 1]$ and $U$ is closed in $(F_{USCG}(X), H_{end})$

**Proof.** The equivalence of statements (i) and (ii) follows immediately from Theorem 4.10. Obviously statement (iii) implies statement (ii).

Now we prove that statement (i) implies statement (iii). Suppose that $U$ is compact. To show that $U(\alpha)$ is compact, we only need to show that $U(\alpha)$ is closed.

Let $\{x_n\}$ be a sequence in $U(\alpha)$ with $x_n \to x$. Suppose that $x_n \in [u_n]_\alpha$ and $u_n \in U$ for $n = 1, 2, \ldots$. Then there exist subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in U$ such that $H_{end}(u_{n_k}, u) \to 0$. By Theorems 3.1 and 2.4, we have that $\limsup_{n \to \infty} [u_{n_k}]_\alpha \subseteq [u]_\alpha$. Hence $x \in [u]_\alpha$, and thus $x \in U(\alpha)$.

We can also obtain that $x \in [u]_\alpha$ from the fact that $H([u_{n_k}]_\alpha, [u]_\alpha) \to 0$.

**Theorem 4.12.** Let $(X, d)$ be a metric space. Then the following statements are equivalent.

(i) $(X, d)$ is complete

(ii) $(F_{USCG}(X), H_{end})$ is complete

**Proof.** (i) $\Rightarrow$ (ii). Let $\{u_n\}$ be a Cauchy sequence in $(F_{USCG}(X), H_{end})$. Then $U = \{u_n, n = 1, 2, \ldots\}$ is total bounded in $(F_{USCG}(X), H_{end})$. So from the proof the sufficiency part of Theorem 4.8, we know that $\{u_n\}$ has a convergent subsequence in $(F_{USCG}(X), H_{end})$, and thus $\{u_n\}$ is convergent in $(F_{USCG}(X), H_{end})$.

(ii) $\Rightarrow$ (i). Let $\{x_n\}$ be a Cauchy sequence in $X$. Note that $H_{end}(\hat{x}, \hat{y}) = \min\{d(x, y), 1\}$ for $x, y \in X$. Then $\{\hat{x}_n\}$ is a Cauchy sequence in $(F_{USCG}(X), H_{end})$, and therefore $\{\hat{x}_n\}$ converges to $u \in F_{USCG}(X)$. Thus there exists an $x \in X$ such that $[u]_\alpha = \{x\}$ for all $\alpha \in [0, 1]$ (i.e. $u = \hat{x}$) and $d(x_n, x) \to 0$.

\[\square\]
5. Characterizations of compactness and completions

The contents in this section are as follows: we give the characterizations of totally bounded sets and compact sets in $(F_{USCB}(X), H_{send})$. Even if $(X, d)$ is complete, $(F_{USCB}(X), H_{send})$ need not be complete. We give a completion of $(F_{USCB}(X), H_{send})$. We also show that $(F_{USCG}(\tilde{X}), H_{end})$ is a completion of $(F_{USCB}(X), H_{end})$, and thus a completion of $(F_{USCG}(X), H_{end})$.

We introduce $P_{USC}(X)$ and $P_{USCB}(X)$ which are subsets of $X \times [0, 1]$.

\begin{align*}
P_{USC}(X) & := \{ u \subseteq X \times [0, 1] : \langle u \rangle_\alpha = \bigcap_{\beta < \alpha} \langle u \rangle_\beta \text{ for all } \alpha \in (0, 1] ; \} \\
P_{USCB}(X) & := \{ u \in P_{USC}(X) : \langle u \rangle_\alpha \in C(X) \text{ for all } \alpha \in [0, 1] \},
\end{align*}

where $\langle u \rangle_\alpha := \{ x : (x, \alpha) \in u \}$ for $u \subseteq X \times [0, 1]$ and $\alpha \in [0, 1]$.

It can be checked that if $u \in P_{USC}(X)$ then $u \in C(X \times [0, 1])$, and that if $u \in P_{USCB}(X)$ then $u \in K(X \times [0, 1])$.

It can also be checked that if $u \in C(X \times [0, 1])$, then $\langle u \rangle_\alpha \in C(X)$ for all $\alpha \in [0, 1]$, and that if $u \in K(X \times [0, 1])$, then $\langle u \rangle_\alpha \in K(X)$ for all $\alpha \in [0, 1]$.

So we can write

\begin{equation*}
P_{USC}(X) = \{ u \in C(X \times [0, 1]) : \langle u \rangle_\alpha = \bigcap_{\beta < \alpha} \langle u \rangle_\beta \text{ for all } \alpha \in (0, 1] \},
\end{equation*}

\begin{equation*}
P_{USCB}(X) = \{ u \in P_{USC}(X) : u \in K(X \times [0, 1]) \}.
\end{equation*}

We can formally define $H_{send}$ and $H_{end}$ on $P_{USC}(X)$

\begin{align*}
H_{send}(u, v) & := H(u, v), \\
H_{end}(u, v) & := H(u, v),
\end{align*}

where $u := u \cup (X \times \{0\})$. Clearly, $H_{send}$ is a metric on $P_{USC}(X)$. However, $H_{end}$ need not be a metric on $P_{USC}(X)$.

Consider the function $f : F_{USC}(X) \to P_{USC}(X)$ given by $f(u) = \text{send } u$. Then

- $f$ is an isometric embedding of $(F_{USC}(X), H_{send})$ in $(P_{USC}(X), H_{send})$.
- $f|_{F_{USCB}(X)}$ is an isometric embedding of $(F_{USCB}(X), H_{send})$ in $(P_{USCB}(X), H_{send})$.

So we can think of $(F_{USC}(X), H_{send})$ as a subspace of $(P_{USC}(X), H_{send})$ and $(F_{USCB}(X), H_{send})$ as a subspace of $(P_{USCB}(X), H_{send})$. 

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Remark 5.1. Clearly, from the above observation, it is natural to discuss the properties of \((F_{USCB}(X), H_{send})\) by treating \((F_{USCB}(X), H_{send})\) as a subspace of \((P_{USCB}(X), H_{send})\), which is a subspace of \((K(X \times [0,1]), H)\).

Obviously, we can also discuss the properties of \((F_{USCG}(X), H_{end})\) by treating \((F_{USCG}(X), H_{end})\) as a subspace of \((C(X \times [0,1]), H)\) (we can think of each \(u \in F_{USCG}(X)\) as its endograph).

We have considered the above views when we submitted [7], if necessary, we will submit the detailed discussion corresponding to the above views.

For \(u \in F_{USC}(X)\), we use \(\overrightarrow{u}\) to denote \(f(u)\).

For \(v \in P_{USC}(X)\), we use \(\overrightarrow{v}\) to denote \(f^{-1}(v')\), where \(v' \in f(F_{USC}(X))\) is given by

\[
\langle v' \rangle_\alpha = \begin{cases} 
\langle v \rangle_\alpha, & \alpha \in (0,1], \\
\cup_{\alpha > 0} \langle v \rangle_\alpha, & \alpha = 0.
\end{cases}
\]

Theorem 5.2. Let \(u\) be an element of \(P_{USC}(X)\) and \(\{u_n\}\) a sequence in \(P_{USC}(X)\). Then \(H_{send}(u_n, u) \to 0\) iff \(H_{end}(u_n, u) \to 0\) and \(H([u_n]_0, [u]_0) \to 0\).

Proof. Let \(u, v\) in \(P_{USC}(X)\). Then

\[
H_{end}(u, v) \leq H_{send}(u, v), \tag{9}
\]

\[
H([u]_0, [v]_0) \leq H_{send}(u, v). \tag{10}
\]

Thus \(H_{send}(u_n, u) \to 0\) implies that \(H_{end}(u_n, u) \to 0\) and \(H([u_n]_0, [u]_0) \to 0\).

Let \((x, \alpha) \in send u\). Clearly, \(d((x, \alpha), send v) \leq \alpha + d(x, [v]_0)\). If \(\alpha > d((x, \alpha), end v)\), then \(d((x, \alpha), send v) = d((x, \alpha), end v)\). Hence

\[
d((x, \alpha), send v) \leq \begin{cases} 
H_{end}(u, v), & \alpha > H_{end}(u, v), \\
\alpha + H([u]_0, [v]_0), & \alpha \in [0,1].
\end{cases}
\]

Thus for \(u, v \in P_{USC}(X)\) with \(H_{end}(u, v) < 1\)

\[
H_{send}(u, v) \leq H_{end}(u, v) + H([u]_0, [v]_0). \tag{11}
\]

(The “=” can be obtained in (11)). So \(H_{end}(u_n, u) \to 0\) and \(H([u_n]_0, [u]_0) \to 0\) imply that \(H_{send}(u_n, u) \to 0\).

\[\square\]

Theorem 5.3. Suppose that \(U\) is a subset of \(F_{USCB}(X)\). Then \(U\) is totally bounded in \((F_{USCB}(X), H_{send})\) if and only if \(U(0)\) is totally bounded in \((X, d)\).
Proof. Necessity. Suppose that $U$ is totally bounded. By (10), $U_0$ is totally bounded in $(K(X), H)$. From Theorem 4.2, this is equivalent to $U(0)$ is totally bounded in $(X, d)$.

Sufficiency. Suppose that $U(0)$ is totally bounded. Then $U(\alpha)$ is totally bounded for each $\alpha \in [0, 1]$. To show that $U$ is totally bounded in $(F_{USCB}(X), H_{send})$, we only need to prove that each sequence in $U$ has a Cauchy subsequence with respect to $H_{send}$.

Let $\{u_n\}$ be a sequence in $U$. Then by Theorem 4.8, $\{u_n\}$ has a Cauchy subsequence $\{v_n\}$ in $(F_{USCB}(X), H_{end})$. From Theorem 4.2, $\{v_n\}$ has a subsequence $\{w_n\}$ such that $\{[w_n]_0\}$ is a Cauchy sequence in $(K(X), H)$. Thus by (11), $\{w_n\}$ is a Cauchy sequence in $(F_{USCB}(X), H_{send})$.

\[\square\]

**Theorem 5.4.** Let $(X, d)$ be a metric space. Then the following statements are equivalent.

(i) $X$ is complete

(ii) $(P_{USCB}(X), H_{send})$ is complete

Proof. (i) $\Rightarrow$ (ii). Let $\{u_n\}$ be a Cauchy sequence in $(P_{USCB}(X), H_{send})$. Note that for each $u, v \in P_{USCB}(X)$, $H_{end}(u, v) = H_{end}(\hat{u}, \hat{v})$. So by (9), $\{\hat{u}_n\}$ is a Cauchy sequence in $(F_{USCG}(X), H_{end})$. From Theorem 4.12, $\{\hat{u}_n\}$ converges to $u \in F_{USCG}(X)$.

By (10), $\{\langle u_n \rangle_0\}$ is a Cauchy sequence in $(K(X), H)$. Thus $\{\langle u_n \rangle_0\}$ converges to $u_0 \in K(X)$.

Set $w \in P_{USCB}(X)$ given by

$$\langle w \rangle_\alpha = \begin{cases} [u]_\alpha, & \alpha > 0, \\ u_0, & \alpha = 0. \end{cases}$$

Thus from Theorem 5.2, $u_n$ converges to $w$ in $(P_{USCB}(X), H_{send})$.

(ii) $\Rightarrow$ (i). Note that $d(x, y) = H_{send}(\tilde{x}, \tilde{y})$. So the desired result follows from the fact that $(X, d)$ is isometric to a closed subspace of $(P_{USCB}(X), H_{send})$.

\[\square\]

**Theorem 5.5.** $(P_{USCB}(\tilde{X}), H_{send})$ is a completion of $(F_{USCB}(X), H_{send})$.

Proof. From Theorem 5.4, $(P_{USCB}(\tilde{X}), H_{send})$ is complete. To show that $(P_{USCB}(\tilde{X}), H_{send})$ is a completion of $(F_{USCB}(X), H_{send})$, we only need to
show that for each \( u \in P_{\text{USCB}}(\tilde{X}) \) and each \( \varepsilon > 0 \), there is a \( w \in F_{\text{USCB}}(X) \) such that \( H_{\text{send}}(u, \overrightarrow{w}) \leq \varepsilon \). To show this is equivalent to show the following affirmations (a) and (b)

(a) For each \( u \in P_{\text{USCB}}(\tilde{X}) \) and each \( \varepsilon > 0 \), there exists a \( v \in F_{\text{USCB}}(\tilde{X}) \) such that \( H_{\text{send}}(u, \overrightarrow{v}) \leq \varepsilon \)

(b) For each \( v \in F_{\text{USCB}}(\tilde{X}) \) and each \( \varepsilon > 0 \), there exists a \( w \in F_{\text{USCB}}(X) \) such that \( H_{\text{send}}(v, w) \leq \varepsilon \)

Let \( u \in P_{\text{USCB}}(\tilde{X}) \). Define \( u_\varepsilon \in F_{\text{USCB}}(\tilde{X}), \varepsilon > 0 \), given by

\[
[u_\varepsilon]_\alpha = \begin{cases} 
\langle u \rangle_\alpha, & \alpha \in (\varepsilon, 1], \\
\langle u \rangle_0, & \alpha \in [0, \varepsilon].
\end{cases}
\]

Then \( H_{\text{send}}(u, \overrightarrow{u_\varepsilon}) \leq \varepsilon \). So affirmation (a) is proved.

Let \( v \in F_{\text{USCB}}(\tilde{X}) \). We can choose a finite subset \( C_0 \) of \( X \) such that \( H(C_0, [v]_0) < \varepsilon \). Define

\[
C_\alpha := \{ x \in C_0 : d(x, [v]_\alpha) \leq \varepsilon \}, \quad \alpha \in (0, 1].
\]

We affirm that \( \{C_\alpha : \alpha \in [0, 1]\} \) has the following properties

(i) \( C_\alpha \neq \emptyset \) for all \( \alpha \in [0, 1] \).

(ii) \( H(C_\alpha, [v]_\alpha) \leq \varepsilon \) for all \( \alpha \in [0, 1] \).

(iii) \( C_\alpha = \bigcap_{\beta < \alpha} C_\beta \) for all \( \alpha \in (0, 1] \).

(iv) \( C_0 = \bigcup_{\alpha > 0} C_\alpha = \bigcup_{\alpha > 0} C_\alpha \).

For each \( y \in [v]_\alpha \), there exists \( z_y \in C_\alpha \) such that \( d(y, z_y) = d(y, C_0) < \varepsilon \). Hence \( z_y \in C_\alpha \) and thus \( C_\alpha \neq \emptyset \). So (i) is true.

To show (ii), we only need to show that \( H(C_\alpha, [v]_\alpha) \leq \varepsilon \) for \( 0 < \alpha \leq 1 \).

Let \( \alpha \in (0, 1] \). From (12), \( H^*(C_\alpha, [v]_\alpha) \leq \varepsilon \). In the following, we show that \( H^*([v]_\alpha, C_\alpha) < \varepsilon \). In fact, for each \( y \in [v]_\alpha \), \( d(y, C_\alpha) \leq d(y, z_y) = d(y, C_0) \) (hence \( d(y, C_\alpha) = d(y, C_0) \)). Thus \( H^*([v]_\alpha, C_\alpha) \leq H([v]_0, C_0) < \varepsilon \). So (ii) is proved.

Let \( \alpha \in (0, 1] \). Clearly \( C_\alpha \subseteq \bigcap_{\beta < \alpha} C_\beta \). By Lemma 2.5, \( \lim_{\beta \to \alpha} H([v]_\alpha, [v]_\beta) = 0 \). So for each \( x \in X \), \( d(x, [v]_\alpha) = \lim_{\beta \to \alpha} d(x, [v]_\beta) \), and hence \( C_\alpha \supseteq \bigcap_{\beta < \alpha} C_\beta \). Thus \( C_\alpha = \bigcap_{\beta < \alpha} C_\beta \). So (iii) is true.

Let \( x \in C_0 \). Then \( d(x, [v]_0) < \varepsilon \). Since \( [v]_0 = \bigcup_{\alpha > 0} [v]_\alpha \), there exists \( \alpha > 0 \) such that \( d(x, [v]_\alpha) < \varepsilon \) (in fact, for each \( x \in X \), \( d(x, [v]_0) = \inf_{\alpha > 0} d(x, [v]_\alpha) \)), and thus \( x \in C_\alpha \). So (iv) is proved.
Set \( w \in F(X) \) given by \([w]_\alpha = C_\alpha \) for all \( \alpha \in [0, 1] \). Then by (i), (iii) and (iv), \( w \in F_{USCB}(X) \). From (ii), we have \( H_{send}(v, w) \leq \varepsilon \). So affirmation (b) is proved.

**Theorem 5.6.** \((F_{USCG}(\tilde{X}), H_{end})\) is a completion of \((F_{USCB}(X), H_{end})\).

**Proof.** From Theorem 4.12, affirmation (b) in the proof of Theorem 5.5 and (9), we only need to show that for each \( u \in F_{USCG}(\tilde{X}) \) and each \( \varepsilon > 0 \), there is a \( v \in F_{USCB}(\tilde{X}) \) such that \( H_{end}(u, v) \leq \varepsilon \).

Let \( u \in F_{USCG}(\tilde{X}) \). Define \( u^\varepsilon \in F_{USCB}(\tilde{X}), \varepsilon > 0 \), given by

\[
[u^\varepsilon]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (\varepsilon, 1], \\ [u]_\varepsilon, & \alpha \in [0, \varepsilon]. \end{cases}
\]

Then \( H_{end}(u, u^\varepsilon) \leq \varepsilon \).

**Corollary 5.7.** \((F_{USCG}(\tilde{X}), H_{end})\) is a completion of \((F_{USCG}(X), H_{end})\).

**Proof.** Since \( F_{USCB}(X) \subseteq F_{USCG}(X) \subseteq F_{USCG}(\tilde{X}) \), the desired result follows from Theorem 5.6.

Before proceeding to our next result we need a couple of definitions.

- \( u \in F_{USC}(X) \) is said to be right-continuous at 0 if for each \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( H([u]_\delta, [u]_0) < \varepsilon \).

- \( U \subseteq F_{USC}(X) \) is said to be equi-right-continuous at 0 if for each \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( H([u]_\delta, [u]_0) < \varepsilon \) for all \( u \in U \).

By Lemma 2.5, for each \( u \in F_{USCB}(X) \), \( u \) is right-continuous at 0.

**Theorem 5.8.** Suppose that \( U \) is a subset of \( F_{USCB}(X) \). Then \( U \) is relatively compact in \((F_{USCB}(X), H_{send})\) if and only if \( U(0) \) is relatively compact in \( X \) and \( U \) is equi-right-continuous at 0.

**Remark 5.9.** Theorem 5.8 is a conclusion in [8].

**Theorem 5.10.** Suppose that \( U \) is a subset of \( F_{USCB}(X) \). Then the following statements are equivalent.
(i) $U$ is compact in $(F_{USCB}(X), H_{send})$

(ii) $U$ is closed in $(F_{USCB}(X), H_{send})$, $U(0)$ is relatively compact in $X$ and $U$ is equi-right-continuous at 0

(iii) $U$ is closed in $(F_{USCB}(X), H_{send})$, $U(0)$ is compact in $X$ and $U$ is equi-right-continuous at 0

**Proof.** The desired result follows immediately from Theorems 4.6 and 5.8.

We have already obtained the results in [7] when we submitted [7]. In this paper we submit the proofs of some conclusions in [7], and add Theorem 5.6 and Corollary 5.7.


