HÖLDER CONTINUITY OF SOLUTIONS TO THE G-LAPLACE EQUATION INVOLVING MEASURES

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ABSTRACT. We establish regularity of solutions to the G-Laplace equation
$$-\text{div} \left( \frac{g(|\nabla u|)}{\nabla u} \nabla u \right) = \mu,$$
where $\mu$ is a nonnegative Radon measure satisfying $\mu(B_r(x_0)) \leq Cr^m$ for any ball $B_r(x_0) \subset \subset \Omega$ with $r \leq 1$ and $m > n - 1 - \delta \geq 0$. The function $g(t)$ is supposed to be nonnegative and $C^1$-continuous in $[0, +\infty)$, satisfying $g(0) = 0$, and for some positive constants $\delta$ and $g_0$, $0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0$, $\forall t > 0$, that generalizes the structural conditions of Ladyzhenskaya-Ural’tseva for an elliptic operator.

1. Introduction.

Let $\Omega$ be an open bounded domain of $\mathbb{R}^n (n \geq 2)$, and $\mu$ a nonnegative Radon measure in $\Omega$ with $\mu(B_r(x_0)) \leq Cr^m$ for some constant $C > 0$ whenever $B_r(x_0) \subset \subset \Omega$. We consider the equation
$$-\Delta_G u = -\text{div} \left( \frac{g(|\nabla u|)}{\nabla u} \nabla u \right) = \mu \quad \text{in} \ D'(\Omega),$$
where $G(t) = \int_0^t g(s)ds$, $g(t)$ is a nonnegative $C^1$ function in $[0, +\infty)$, satisfying $g(0) = 0$ and the following structural condition
$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall t > 0, \quad \delta, g_0 \text{ are positive constants}.$$

The structural conditions on $g$ was introduced by Lieberman in 1991, which is a natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations (see [10]). The conditions of $g$ imply that the operator $\Delta_G$ includes not only the $p$-Laplace operator $\Delta_p$ where $g(t) = tp^{p-1}$ and $\delta = g_0 = p - 1$, but also the case of a variable exponent $p = p(t) > 0$:
$$-\Delta_G u = -\text{div} \left( |\nabla u|^{p(t)-2} \nabla u \right),$$
for which (2) holds if $\delta \leq t(ln t)p'(t) + p(t) - 1 \leq g_0$ for all $t > 0$. Another typical example of $g$ is $g(t) = p\log(at+b)$ with $p, a, b > 0$ where in this case $\delta = p$ and $g_0 = p + 1$. Many other examples can be found in [2, 3, 6] etc.

Under assumption (2), $G$ is an increasing $C^2$ convex function, which is an $N$-function satisfying the so called $\Delta_2$-condition. Thus our class of operators will be

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considered in the setting of Orlicz spaces. We recall the definitions of Orlicz and Orlicz-Sobolev spaces together with their respective norms (see [1])

\[
L^G(\Omega) = \{ u \in L^1(\Omega) : \int_{\Omega} G(|u(x)|) \, dx < +\infty \},
\]

\[
\|u\|_{L^G(\Omega)} = \inf \left\{ k > 0 : \int_{\Omega} G \left( \frac{|u(x)|}{k} \right) \, dx \leq 1 \right\},
\]

\[
W^{1,G}(\Omega) = \{ u \in L^G(\Omega) : \nabla u \in L^G(\Omega) \},
\]

\[
\|u\|_{W^{1,G}(\Omega)} = \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}.
\]

Under the assumption (2), \(W^{1,G}(\Omega)\) is a reflexive and separable Banach space (see [1]).

We shall call a solution of (1) any function \(u \in W^{1,G}_\text{loc}(\Omega)\) that satisfies

\[
\int_{\Omega} \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \cdot \phi \, dx = \int_{\Omega} \phi \, d\mu \quad \forall \phi \in D(\Omega).
\]

If \(\mu \equiv 0\) in a domain \(D \subset \Omega\), we say that \(u\) is \(G\)-harmonic in \(D\).

We now introduce the regularity of the related elliptic equations involving measures. In 1994, Kilpeläinen considered the situation of the \(p\)-Laplace operator and proved that if \(\mu\) satisfies \(\mu(B_r) \leq C r^{n-p+\alpha(p-1)}\) for some positive constants \(C\) and \(\alpha \in (0,1]\), then any solution of the \(p\)-Laplace equation

\[
-\Delta_p u = -\text{div} (|\nabla u|^{p-2} \nabla u) = \mu,
\]

is \(C^{0,\beta}_{\text{loc}}\)-continuous for each \(\beta \in (0,\alpha)\) (see [7]). This result was improved by Kilpeläinen and Zhong in 2002, showing that if each solution of (3) is in fact Hölder continuous with the same exponent \(\alpha\) as the one in the assumption \(\mu(B_r) \leq C r^{n-p+\alpha(p-1)}\) (see [8]). In 2010, the \(p\)-Laplace problem (3) was extended by Lyaghfouri to the case with variable exponents, i.e., considering

\[
-\text{div} (|\nabla u|^{p(x)-2} \nabla u) = \mu.
\]

Under certain assumptions on the function \(p(x)\) and the assumption \(\mu(B_r) \leq C r^{n-p(x)+\alpha(p(x)-1)}\) for some positive constants \(C\) and \(\alpha \in (0,1]\), the author proved that any bounded solution of (4) is \(C^{0,\alpha}_{\text{loc}}\)-continuous with the same exponent \(\alpha\) (see [11]).

When focusing on the problem governed by \(G\)-Laplacian, if \(\mu(B_r(x_0)) \leq C r^m\) with \(m \in [n-1, n)\), Challal and Lyaghfouri proved that any solution of (1) is \(C^{0,\alpha}_{\text{loc}}\)-continuous with \(\alpha = \frac{m-n+1+\delta}{1+\gamma_0}\) (see [3]). Particularly, if \(m = n-1\), then any solution is \(C^{0,\alpha}_{\text{loc}}\)-continuous for any \(\alpha \in (0, \frac{\delta}{\gamma_0})\) (see Theorem 3.3 in [3]). In 2011, these regularities were improved by Challal and Lyaghfouri in [5], showing that any local bounded solution of (1) is \(C^{0,\alpha}_{\text{loc}}\)-continuous for any \(\alpha \in \left(0, \frac{m-n+1+\delta}{\gamma_0}\right)\) provided that \(m > n - 1 - \delta\). Note that under the assumption of non-decreasing monotonocity on \(\frac{g(t)}{t}\), Zheng, Feng and Zhang obtained local \(C^{1,\alpha}_{\text{loc}}\)-continuity of solutions for \(m > n\) and local Hölder continuity with small exponents for some \(m < n\) in 2015 (see [14]).

In this paper, we continue the work of Challal, Lyaghfouri and Zheng et al. by improving the regularity of solutions of the equation (1). Particularly, we can prove the \(C^{0,\alpha}_{\text{loc}}\)-continuity of solutions for any \(\alpha \in (0,1]\) if \(m = n-1\). More precisely, for any \(m > n - 1 - \delta\) and without any monotonicity assumption on \(\frac{g(t)}{t}\), we have the following result.
Theorem 1.1. Assume that $\mu$ satisfies (1) with $m > n - 1 - \delta \geq 0$. Then we have

(i) If $m > n$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, \min\{\frac{\sigma}{1 + m}, \frac{m-n}{2(1+mg)}\})$, where $\sigma$ is the same as in Lemma 2.4.

(ii) If $m \in [n-1, n)$, then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, 1)$.

Remark 1. In [7], the author proved for the $p$-Laplacian problem that $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0, 1)$ provided $m = n - 1$. In this paper we not only improve the results of [3, 5] and [14], but also extend the problem in [7] to general equations which governed by a large class of degenerate and singular elliptic operators.

2. Preliminary.

In this section, we state some auxiliary results which will be used throughout this paper. We begin with some properties of the function $G$.

Lemma 2.1 ([13, Lemma 2.1, Remark 2.1]). Function $G$ has the following properties:

(G1) $G$ is convex and $C^2$.

(G2) $\frac{g(t)}{t^{1+mg}} \leq G(t) \leq t g(t)$, for all $t \geq 0$.

(G3) $\min\{s^{g+1}, s^{g_0+1}\} \frac{G(t)}{t^{1+g_0}} \leq G(st) \leq (1 + g_0) \max\{s^{g+1}, s^{g_0+1}\} G(t)$.

(G4) $G(a + b) \leq 2^{g_0}(1 + g_0)(G(a) + G(b))$ for all $a, b > 0$.

For much more properties of $G$ and problems governed by the operator $\Delta_G$, please see [2, 3, 4, 5, 6, 13, 14, 15, 16] etc.

The following lemmas are some properties of $G$-harmonic functions. Throughout this paper, without special states, by $B_R$ and $B_r$ we denote the balls contained in $\Omega$ with the same center. Moreover, $B_r \subset B_R \subset \subset \Omega$.

Lemma 2.2 ([13, Theorem 2.3]). Assume $u \in W^{1,G}(\Omega)$. Let $h$ be a weak solution of

$$\Delta_G h = 0 \quad \text{in } B_R, \quad h - u \in W^{1,G}_0(B_R),$$

then

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) dx \geq C \left( \int_{A_2} G(|\nabla u - \nabla h|) dx + \int_{A_1} \frac{g(|\nabla u|)}{|\nabla u|} |\nabla u - \nabla h|^2 dx \right),$$

where $A_1 = \{x \in B_R; |\nabla u - \nabla h| \leq 2|\nabla u|\}$, $A_2 = \{x \in B_R; |\nabla u - \nabla h| > 2|\nabla u|\}$ and $C = C(\delta, g_0) > 0$.

Lemma 2.3 ([13, Lemma 2.7]). Let $h \in W^{1,G}(\Omega)$ be a weak solution of $\Delta_G h = 0$. Then $h \in C^{1,\alpha}(\Omega)$. Moreover, there exists $C = C(n, \delta, g_0) > 0$ such that for every ball $B_r \subset \subset \Omega$ and every $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, ||h||_{L^\infty(B_{2r}(x_0))}) > 0$ such that

$$\int_{B_r} G(|\nabla h|) dx \leq Cr^\lambda.$$

Let $(u)_r = \frac{1}{|B_r|} \int_{B_r} u dx$ be the average value of $u$ on the ball $B_r$, we have
Lemma 2.4 (Comparison with $G$-harmonic functions [14, Lemma 3.1]). Assume $u \in W^{1,G}(B_R)$. Let $h \in W^{1,G}(B_R)$ be a weak solution of $\Delta_G h = 0$ in $B_R$. Then there exists $\sigma \in (0, 1)$ and $C = C(n, \delta, g_0) > 0$ such that for each $0 < r \leq R$, there holds
\[
\int_{B_r} G(|\nabla u - (\nabla u)_r|) \, dx \leq C \left( \frac{r}{R} \right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) \, dx + C \int_{B_R} G(|\nabla u - \nabla h|) \, dx.
\]

Lemma 2.5 ([9, Lemma 2.7]). Let $\phi(s)$ be a non-negative and non-decreasing function. Suppose that $\phi(r) \leq C_1 \left( \frac{r}{R} \right)^{\alpha} \phi(R) + C_1 R^\beta$, for all $r \leq R \leq R_0$, with $\alpha, \beta$ and $C_1$ positive constants. Then, for any $\tau < \min\{\alpha, \beta\}$, there exists a constant $C_2 = C_2(C_1, \alpha, \beta, \tau)$ such that for all $r \leq R \leq R_0$ we have
\[
\phi(r) \leq C_2 r^\tau.
\]

3. Proof of Theorem 1.1.

Lemma 3.1. Assume $u \in W^{1,G}(\Omega)$. Let $B_R \subset \subset \Omega$ and $h \in W^{1,G}(B_R)$ be a weak solution of
\[
\Delta_G h = 0 \quad \text{in } B_R, \quad h - u \in W_0^{1,G}(B_R).
\]
Then for any $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|u\|_{L^\infty(B_{2R/3})}) > 0$ such that
\[
\int_{B_R} G(|\nabla u - \nabla h|) \, dx \leq CR^m + CR^{\frac{m+\lambda}{\lambda}},
\]
where $\lambda$ is the same as in Lemma 2.3.

Proof. Firstly, convexity of $G$ gives
\[
\int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx \leq \int_{B_R} \frac{g(|\nabla u|)}{|\nabla u|} \nabla u (\nabla u - \nabla h) \, dx = \int_{B_R} (u - h) \, d\mu \quad (5)
\]
\[
\leq C \mu(B_R) \leq CR^m, \quad (6)
\]
where we used the boundedness of $u$ which forces $h$ to be bounded too.

Let be $A_1$ and $A_2$ be defined as in Lemma 2.2. By Lemma 2.2, there exists a constant $C = C(\delta, g_0) > 0$ such that
\[
\int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx \geq C \int_{A_2} G(|\nabla u - \nabla h|) \, dx \quad (7)
\]
and
\[
\int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx \geq C \int_{A_1} \frac{g(|\nabla u|)}{|\nabla u|} |\nabla u - \nabla h|^2 \, dx. \quad (8)
\]
By \((G_2)\), \(\frac{G(t)}{t}\) is increasing in \(t > 0\). It follows from \((G_2)\), \((G_3)\), (6), (8) and Lemma 2.2 that

\[
\int_{A_1} G(|\nabla u - \nabla h|) \, dx = \int_{A_1} \frac{G(|\nabla u - \nabla h|)}{|\nabla u - \nabla h|} |\nabla u - \nabla h| \, dx \\
\leq \int_{A_1} \frac{G(2|\nabla u|)}{2|\nabla u|} |\nabla u - \nabla h| \, dx \\
\leq C \int_{A_1} \frac{G(|\nabla u|)}{|\nabla u|} |\nabla u - \nabla h| \, dx \\
= C \int_{A_1} \frac{\sqrt{G(|\nabla u|)}}{|\nabla u|} |\nabla u - \nabla h| \cdot \sqrt{G(|\nabla u|)} \, dx \\
\leq C \left( \int_{A_1} \frac{G(|\nabla u|)}{|\nabla u|^2} |\nabla u - \nabla h|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{A_1} G(|\nabla u|) \, dx \right)^{\frac{1}{2}} \\
\leq C \left( \int_{A_1} \frac{g(|\nabla u|) |\nabla u|}{|\nabla u|^2} |\nabla u - \nabla h|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{A_1} G(|\nabla u|) \, dx \right)^{\frac{1}{2}} \\
= C \left( \int_{A_1} \frac{g(|\nabla u|) |\nabla u - \nabla h|^2 \, dx}{|\nabla u|^2} \right)^{\frac{1}{2}} \left( \int_{A_1} G(|\nabla u|) \, dx \right)^{\frac{1}{2}} \\
\leq C \left( \int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx \right)^{\frac{1}{2}} \left( \int_{B_R} G(|\nabla u|) \, dx \right)^{\frac{1}{2}} \\
= C \left( \int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx \right)^{\frac{1}{2}} \cdot \left( \int_{B_R} (G(|\nabla u|) - G(|\nabla h|) + G(|\nabla h|)) \, dx \right)^{\frac{1}{2}} \\
\leq C \int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx \\
+ C \left( \int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx \right)^{\frac{1}{2}} \left( \int_{B_R} G(|\nabla h|) \, dx \right)^{\frac{1}{2}},
\]

where in the last inequality but one we used \((a+b)^\gamma \leq a^\gamma + b^\gamma\) for any \(a \geq 0, b \geq 0\) and \(\gamma \in (0, 1)\). By (7) and (9), we have

\[
\int_{B_R} G(|\nabla u - \nabla h|) \, dx = \int_{A_2} G(|\nabla u - \nabla h|) \, dx + \int_{A_1} G(|\nabla u - \nabla h|) \, dx \\
\leq C \int_{B_R} (G(|\nabla u|) - G(|\nabla h|)) \, dx + CR^m + CR^{\frac{m+\lambda}{2}} \\
\leq CR^m + CR^{\frac{m+\lambda}{2}}.
\]

1 Proof of Theorem 1.1. Let \(h\) be a \(G\)-harmonic function in \(B_R\) that agrees with \(u\) on the boundary, i.e.,

\[
\text{div} \frac{g(|\nabla h|)}{|\nabla h|} \nabla h = 0 \text{ in } B_R \quad \text{and} \quad h - u \in W^{1,\bar{G}}_0(B_R).
\]
By Lemma 2.4 and Lemma 3.1, for any \( r \leq R \) there holds
\[
\int_{B_r} G(|\nabla u - (\nabla u)_r|) \, dx \\
\leq C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) \, dx + C\int_{B_R} G(|\nabla u - \nabla h|) \, dx \\
\leq C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) \, dx + CR^m + CR^{\frac{m+\lambda}{2}},
\]
where \( \lambda \) is an arbitrary constant in \((0, n)\).

(i) If \( m > n \), then we have
\[
\int_{B_r} G(|\nabla u - (\nabla u)_r|) \, dx \leq C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|) \, dx + CR^{\frac{m+\lambda}{2}}.
\]
Since \( m > n \) and \( \lambda \) is an arbitrary constant in \((0, n)\), one may choose \( \lambda \) satisfying \( \frac{m+\lambda}{2} > n \). In view of Lemma 2.5, we conclude that for any \( \tau < \min\{\sigma, \frac{m+\lambda}{2} - n\} \) there holds
\[
\int_{B_r} G(|\nabla u - (\nabla u)_r|) \, dx \leq Cr^{n+\tau}, \quad \forall r \leq R. \tag{10}
\]
Now we claim that
\[
\int_{B_r} |\nabla u - (\nabla u)_r| \, dx \leq Cr^{n+\frac{\tau}{1+\tau}}, \quad \forall r \leq R. \tag{11}
\]
Indeed, for \( r \) satisfying \( r^{-n} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx \leq r^{\frac{\tau}{1+\tau}} \), (11) holds with \( C = 1 \). Now for \( r \) satisfying \( r^{-n} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx > r^{\frac{\tau}{1+\tau}} \), we infer from the increasing monotonicity of \( \frac{G(t)}{t^{\frac{\tau}{1+\tau}}} \) in \( t > 0 \),
\[
\frac{G\left(r^{-n} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx \right)}{r^{-n} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx} \geq \frac{G\left(r^{\frac{\tau}{1+\tau}} \right)}{r^{\frac{\tau}{1+\tau}}}.
\]
It follows from \((G_2)\) and \((G_3)\)
\[
\int_{B_r} |\nabla u - (\nabla u)_r| \, dx \leq \frac{r^{n+\frac{\tau}{1+\tau}}}{G\left(r^{\frac{\tau}{1+\tau}} \right)} G\left(r^{-n} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx \right) \\
\leq \frac{C r^{n+\frac{\tau}{1+\tau}}}{r^\tau G(1)} G\left(r^{-n} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx \right) \\
\leq \frac{C r^{n+\frac{\tau}{1+\tau}}}{r^\tau g(1)} G\left(r^{-n} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx \right). \tag{12}
\]
Note that convexity of \( G \) and (10) implies that
\[
G\left(\frac{1}{|B_r|} \int_{B_r} |\nabla u - (\nabla u)_r| \, dx \right) \leq \frac{1}{|B_r|} \int_{B_r} G(|\nabla u - (\nabla u)_r|) \, dx \leq Cr^\tau. \tag{13}
\]
By \((G_3)\), (12) and (13), one may get
\[
\int_{B_r} |\nabla u - (\nabla u)_r| \, dx \leq Cr^{n+\frac{\tau}{1+\tau}},
\]
where \( C \) depends only on \( g(1), g_0 \) and the volume of the unit ball. Now we have proven that (11) holds for any \( r \leq R \). Thus \( u \in C^{1, \frac{\tau}{1+\tau}}_{loc} (\Omega) \) by Campanato’s
embedding Theorem. Due to the arbitrary of \( \lambda \in (0, n) \), we can conclude (i) of Theorem 1.1 by letting \( \lambda \to n \).

(ii) If \( m \in [n-1, n] \), we only prove for \( m = n-1 \) due to the fact \( \mu(B_r) \leq C r^{m} \leq C r^{n-1} \) with small \( r \). By (G4), Lemma 2.3 and Lemma 3.1, we infer

\[
\int_{B_r} G(|\nabla u|)dx \leq C \int_{B_r} G(|\nabla u - \nabla h|)dx + C \int_{B_r} G(|\nabla h|)dx \leq C r^m + C r^{m+\lambda} + C r^\lambda \leq C r^m,
\]

where in the last inequality we let \( n > \lambda > n-1 = m \).

We claim that for any \( r \leq R < 1 \) with \( B_R \subset \subset \Omega \) and some positive constant \( C \) independent of \( r \), there holds

\[
\int_{B_r} |\nabla u|dx \leq C r^{n-1+\alpha_0}, \tag{14}
\]

with some \( \alpha_0 \in (0, 1) \).

Indeed, for \( r \leq R \) satisfying

\[
r^{-n+1-\alpha_0} \int_{B_r} |\nabla u|dx \leq 1, \tag{15}
\]

(14) holds with \( C = 1 \). For \( r \leq R \) satisfying

\[
r^{-n+1-\alpha_0} \int_{B_r} |\nabla u|dx \geq 1,
\]

due to the increasing monotonicity of \( F(t) = G(t) - G(1)t \) in \( t \geq 1 \), it follows

\[
G\left(r^{-n+1-\alpha_0} \int_{B_r} |\nabla u|dx \right) \geq G(1) \cdot r^{-n+1-\alpha_0} \int_{B_r} |\nabla u|dx.
\]

Then we have

\[
\int_{B_r} |\nabla u|dx \leq C r^{n-1+\alpha_0} (r^{1-\alpha_0})^{1+\delta} G\left(r^{-n} \int_{B_r} |\nabla u|dx \right) \leq C r^{n-1+\alpha_0} \cdot (r^{1-\alpha_0})^{1+\delta} \frac{1}{|B_r|} \int_{B_r} G(|\nabla u|)dx \leq C r^{n-1+\alpha_0 + (1-\alpha_0)(1+\delta)} \cdot r^{-n} \cdot r^m = C r^{n-1+\alpha_0 + (1-\alpha_0)(1+\delta) + m-n}. \tag{16}
\]

Combining (15) and (16), we may choose \( \alpha_0 = \alpha_0 + (1-\alpha_0)(1+\delta) + m-n \), i.e., \( \alpha_0 = 1 - \frac{n-m}{1+\delta} \) such that (14) holds for all \( r \leq R \).

For \( m = n-1 \), we conclude that \( u \in C_{loc}^{0,\alpha_0}(\Omega) \) by Morrey Theorem (see page 30, [12]) with \( \alpha_0 = \frac{\delta}{1+\delta} \).
Finally, we have
\[ \alpha_k = \frac{\delta}{1+\delta} + \frac{\alpha_{k-1}}{1+\delta}, \]
where osc\((u, B_r) = \sup_{B_r} u - \inf_{B_r} u\). Arguing as (14), we get \( u \in C^{\alpha_1}_{loc}(\Omega) \) with
\[ \alpha_1 = 1 - \frac{n-(m+\alpha_0)}{1+\delta} = \frac{\delta}{1+\delta} + \frac{\alpha_0}{1+\delta}. \]
Repeating this process, we get \( u \in C^{\alpha_k}_{loc}(\Omega) \) with
\[ \alpha_k = \frac{\delta}{1+\delta} + \frac{\alpha_{k-1}}{1+\delta}, \]
Finally, we have \( \alpha_k = \frac{\alpha_0}{(1+\delta)^k} + \delta\sum_{j=1}^{k} \frac{1}{(1+\delta)^j} \), which leads to \( \lim_{k \to \infty} \alpha_k = 1 \), and the result follows.

(iii) If \( n-1 - \delta < m < n-1 \), checking the proof and repeating the process as above, we may get \( \alpha_0 = 1 - \frac{n-m}{1+\delta} \), \( \alpha_1 = \frac{1+\delta+m-n}{1+\delta} + \frac{\alpha_0}{1+\delta} \), ..., \( \alpha_k = \frac{1+\delta+m-n}{1+\delta} + \frac{\alpha_{k-1}}{1+\delta} \).
Finally, one has \( u \in C^{\alpha_{loc}}_{\alpha}(\Omega) \) for any \( \alpha \in \left(0, \frac{1+\delta+m-n}{1+\delta}\right) \). □

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