A Note on Elliptic Coordinates

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Abstract. Explicit equations are obtained to convert Cartesian coordinates to elliptic coordinates, based on which an elliptic-coordinate function can be readily mapped on a uniform Cartesian mesh.

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1. Introduction

The elliptical coordinate system \((\xi, \eta)\) as a two-dimensional orthogonal coordinate system has many dynamical and engineering applications, such as Kirchhoff vortex solution \((\text{Lamb 1939})\), insect aerodynamics \((\text{Wang 2000})\) and hydrodynamic wave diffraction \((\text{Chatjigeorgiou and Mavrakis 2012})\). Its coordinate lines are confocal ellipses and hyperbolae and the transformation from elliptic to Cartesian coordinates is given by

\[
\begin{align*}
    x &= c \cosh(\xi) \cos(\eta) \\
    y &= c \sinh(\xi) \sin(\eta) \\
    c^2 &= a^2 - b^2 \\
    \xi &\geq 0, \quad 0 \leq \eta < 2\pi
\end{align*}
\]  

(1)

where \(a\) and \(b\) denote the semi-major and semi-minor axes of the ellipse and \(c\) is the elliptical eccentricity (Figure 1).

![Coordinate lines for an elliptic coordinate system with \(a = 1, b = 0.5\).](image)

Figure 1. Coordinate lines for an elliptic coordinate system with \(a = 1, b = 0.5\).
In the mean time, an explicit equation to transform from Cartesian to elliptic coordinates has not been found in the existing literature. Such a functional conversion is useful for mapping an elliptic-coordinate solution such as the streamfunction of Kirchhoff vortex.

2. Cartesian to elliptic coordinates

In order to invert the functional relation (1), we first eliminate $\xi$ and have

$$\frac{x^2}{\cos^2(\eta)} - \frac{y^2}{\sin^2(\eta)} = c^2$$

which means curves of constant $\eta$ are hyperbolae. The focus distance is $c$ and the eccentricity is $e = \sec(\eta)$.

Let $p = \sin^2(\eta)$, we have

$$\frac{x^2}{1 - p} - \frac{y^2}{p} = c^2$$

which becomes

$$c^2 p^2 + (x^2 + y^2 - c^2) p - y^2 = 0 \quad (2)$$

Then eliminating $\eta$ from (1) we have

$$\frac{x^2}{\cosh^2(\xi)} + \frac{y^2}{\sinh^2(\xi)} = c^2$$

It shows that curves of constant $\xi$ are ellipses. The focus distance is $c$ and the eccentricity is $e = \cosh^{-1}(\xi)$.

Let $q = -\sinh^2(\xi)$, we have
\[
\frac{x^2}{1-q} - \frac{y^2}{q} = c^2
\]

which leads to

\[
c^2 q^2 + (x^2 + y^2 - c^2)q - y^2 = 0
\]

(3)

It is essentially the same as (2). Therefore \((p, q)\) constitute the two roots of a quadratic equation. Since \(0 \leq p \leq 1, q \leq 0\), we have \(p \geq q\), and the two roots are

\[
p = \frac{-B + \sqrt{B^2 + 4c^2y^2}}{2c^2}
\]

\[
q = \frac{-B - \sqrt{B^2 + 4c^2y^2}}{2c^2}
\]

(4)

in which \(B = x^2 + y^2 - c^2\).

From the definition of \(p\) we obtain

\[
\eta_0 = \arcsin(\sqrt{p})
\]

(5)

It has four cases depending on which quadrant the Cartesian point \((x, y)\) is located, i.e.,

\[
\begin{align*}
\eta &= \eta_0, \quad x \geq 0, y \geq 0 \\
\eta &= \pi - \eta_0, \quad x < 0, y \geq 0 \\
\eta &= \pi + \eta_0, \quad x \leq 0, y < 0 \\
\eta &= 2\pi - \eta_0, \quad x > 0, y < 0
\end{align*}
\]

(6)

Based on the definition of \(q\), we can solve \(\xi\) from quadratic equation

\[
e^{4\xi} + (4q - 2)e^{2\xi} + 1 = 0
\]

which has two roots
\[ e^{2\xi} = 1 - 2q \pm 2\sqrt{q^2 - q} \]

Since \( q \leq 0 \), both roots are real and denoted as \((\xi_1, \xi_2)\). They clearly satisfy \( e^{2\xi_1} \cdot e^{2\xi_2} = 1 \), which leads to \( \xi_2 = -\xi_1 < 0 \). Because in elliptical coordinates only non-negative \( \xi \) value is considered, we obtain

\[ \xi = \frac{1}{2} \ln(1 - 2q + 2\sqrt{q^2 - q}) \]  \hspace{1cm} (7)

Eqs.(4-7) are explicit equations to derive elliptic coordinates from Cartesian grid. They can easily be realized via computation softwares such as Matlab.

### 3. Application to Kirchhoff vortex

Kirchhoff vortex is a rotating elliptical region of uniform vorticity \( \omega \) embedded in an irrotational ideal fluid (Kirchhoff 1877). It is the simplest example of nonsmooth weak solutions to the Euler equations and has a discontinuity of vorticity across its elliptical boundary

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

where \( a \) and \( b \) are semi-major and semi-minor axes, respectively. The vortex rotates with constant angular velocity

\[ \Omega = \frac{ab}{(a+b)^2} \omega \]

From Lamb (1939, Act.159), the streamfunction outside the core is
\[
\psi = \frac{1}{4} \Omega (a + b)^2 e^{-2\xi} \cos(2\eta) + \frac{1}{2} \omega ab \xi
\]  

(8)

The instantaneous streamlines in a unsteady flow are given by the curves \( \psi = \text{const} \). In a rotating frame with angular velocity \( \Omega \), Kirchhoff vortex becomes steady and its streamfunction outside the elliptic core is related to the inertial-frame streamfunction (8) by

\[
\psi_R = \psi - \frac{1}{2} \Omega [x(\xi, \eta)^2 + y(\xi, \eta)^2]
\]

(9)

From Eq.(9) it is easy to make a conformal mapping of steady streamfunction on uniform mesh of elliptic coordinates (Figure 2).

![Figure 2. Mapping of streamfunction (9) on a uniform mesh of \((\xi, \eta)\), with \(a = 1, b = 0.5, \omega = 10\).](image)

In order to view the flow field in physical space, we now use the conversion equations (4-7) to plot streamfunction (9) on a uniform Cartesian mesh. As revealed in Figure 3, the elliptical core of Kirchhoff
vortex induces two irrotational eddies that rotate in opposite directions.

![Streamfunction mapping](image)

Figure 3. Mapping of streamfunction (9) on a uniform mesh of \((x, y)\), with \(a = 1, b = 0.5, \omega = 10\).

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**References**


Matlab mfile scripts

function [zeta,eta]=xy2el(x,y,a,b)
%-------------------------------------------------------------------------
% Conversion from Cartesian mesh (x,y) to elliptic coordinates (zeta,eta)
% zeta (>=0), eta [0 2pi)
% a -- semi-major axis
% b -- semi-minor axis
% c -- elliptic eccentricity
% (x,y,zeta,eta) same-size matrix
% Author: Che Sun (CAS, 2016)
% Source: http://eprint.las.ac.cn/abs/201611.00721
%c2=(a^2-b^2);c=2*c2;
x2=x.^2; y2=y.^2;
B=x2+y2-c2;
del2=B.^2+2*c*y2; del=sqrt(del2);
q=(-B+del)/c; q=sqrt(q); et0=asin(q);
eta=x;
i=find(x>=0&y>=0); eta(i)=et0(i);
i=find(x<0&y>=0); eta(i)=pi-et0(i);
i=find(x<=0&y<0); eta(i)=pi+et0(i);
i=find(x>0&y<0); eta(i)=2*pi-et0(i);
p=(-B-del)/c;
del2=p.^2-p; del=sqrt(del2);
zeta=log(1-2*p+2*del)/2;% only keep the positive root

function [x,y]=el2xy(zeta,eta,a,b)
%-------------------------------------------------------------------------
% Conversion from elliptic coordinates (zeta,eta) to Cartesian coordinates
% zeta (>=0), eta [0 2pi)
% a -- semi-major axis
% b -- semi-minor axis
% c -- elliptic eccentricity
% (x,y,zeta,eta) same-size matrix
% Author: Che Sun (CAS, 2016)
% Source: http://eprint.las.ac.cn/abs/201611.00721
%c2=a^2-b^2; c=sqrt(c2);
x=c*cosh(zeta).*cos(eta);
y=c*sinh(zeta).*sin(eta);